Decision Procedures for flat set-theoretical syllogistics.I. General union, powerset and singleton operators

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Abstract

In this paper we show that a class of unquantified multi-sorted set-theoretic formulae involving the notions of powerset, general union, and singleton has a solvable satisfiability problem. We exhibit a normalization procedure that given a model for a formula in our theory, it produces a simpler and *a priori* bounded model whose cardinality depends solely on the size of the given formula.

Key words: Automated Deduction, Decidability, Symbolic Computation.

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1 Introduction

In this paper we investigate the satisfiability problem for the class $3LSSPU$ of multi-sorted set-theoretic formulae where singleton, powerset and general union operators can occur.

Our results relate to the ongoing research work in the field of *Computable Set Theory.* A comprehensive description of this subject and most of the results of the first decade of research can be found in [CFO89]. Briefly, the goals of such a research are twofold:

(1) to have a clear understanding of what is decidable and what is not;

(2) to produce a large collection of satisfiability tests so that common deduction steps can be mechanized.

Satisfiability tests (and, dually, validity tests) have been already provided for numerous classes of unquantified and quantified formulae of set theory. We are in the process of implementing them into an effective system for the verification of the correctness of computer programs (written in set-oriented programming languages such as those in the SETL family) and semi-automatic checking of proofs of mathematical theorems.

The basic set-theoretic language that has been considered is MLS (an acronym for Multi-Level Syllogistic), i.e. the unquantified language of set theory with $=$, $\subseteq$, and $\in$ as predicate symbols, $\cup$, $\cap$, and $\setminus$ as function symbols, and the propositional connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$. A decision test for MLS has been first given in [FOS80].

Subsequently, many extensions of the language MLS with various other set operators and predicates have also been shown to have a solvable satisfiability problem. Also, some classes of quantified set-theoretic formulae properly extending MLS have been shown decidable.

In two recent Ph.D. Theses at New York University, the focus of research has been centered on one hand (see [Pol90]) on enhancing standard theorem proving techniques, such as resolution, by combining them with satisfiability tests for classes of formulae known to have a solvable satisfiability problem; on the other hand, on proving that for specific mathematical fields, such as general topology (see [Cut91]), complete satisfiability tests for certain classes of formulae exist and in many instances allow a more feasible approach to the problem of automating and checking proofs.

In the latter thesis, the formalization of topological and specific set-theoretic concepts has led to the investigation of the satisfiability problem for classes of multi-sorted set-theoretic formulae, i.e. formulae where the variables of the underlying language can range over the disjoint homogeneous classes of *individuals, sets of individuals, sets of sets of individuals,* etc. We will refer to such classes of formulae as *multi-sorted flat syllogistics*, as opposed to the more studied *one-sorted multi-level syllogistics* in which variables can range over the Von Neumann universe of sets, i.e. sets which are well founded over the empty set and characterized by Zermelo-Fraenkel axioms (see [Jec78]).

In this paper, we will continue the investigation in the multi-sorted direction and, specifically, we will give a complete satisfiability test for a multi-sorted flat syllogistic, denoted by $3LSSPU$, which involves powerset, general union, and singleton.

We recall that the corresponding one-sorted class of formulae is still not known to have a solvable satisfiability problem. Therefore the present result can be seen as the first contribution towards the solution of the satisfiability problem for MLS extended with powerset, general union and singleton operators.

Partial solutions can be found in [Can91] and in [CFS87] where, respectively, the satisfiability of MLS with powerset and singleton and that of MLS with general union are established.

2 Preliminary definitions

Let us now introduce some basic definitions.

Let S be a nonempty set, whose elements are *individuals* with no internal structure. Recursively, define

$$\begin{aligned} pow^0(S) &= S \\ pow^1(S) &= pow(S) = \{S' | S' \subseteq S\} \\ pow^k(S) &= pow(pow^{k-1}(S)) \quad \text{for } k > 0 \end{aligned}$$

Moreover, for any $k > 1$ and any $T \in pow^k(S)$ define

$$\begin{aligned} Un^0(T) &= T \\ Un^1(T) &= Un(T) = \bigcup_{T' \in T} T' \\ Un^h(T) &= Un(Un^{h-1}(T)) \quad \text{for } h < k \end{aligned}$$

In view of the above definitions, it is natural to associate to each set $T \in pow^k(S)$ the value k as its *sort*.

By *Two Level Syllogistic (2LS)* we mean the two sorted unquantified set-theoretic class of formulae obtained as the propositional closure of atoms of type

- $X = Y \cup Z$, $X = Y \setminus Z$ $X = Y \cap Z$;
- $x \in X$ $x = y$ $X = Y$,

where capital letters denote set variables and small letters denote element variables. Then a given a formula φ of $2LS$ is said to be *satisfiable* if there exists a nonempty set S and an assignment M (mapping element variables into elements of S and set variables into subsets of S) which makes the formula φ true, when the symbols of $2LS$ are given their standard meaning. An efficient satisfiability test for this class of formulae has been given in [FO78].

Subsequentely, in [CC90a] it was proven that $2LS$ extended with the singleton operator $X = \{x\}$ and the cartesian product operator $X = Y \times Z$[1] has also a solvable satisfiability problem.

The class of formulae $2LS$ can be naturally extended to the class nLS which contains n different sorts of variables and whose atoms are of the following types

- $A^i = \emptyset$, $A^i \in B^{i+1}$ $A^i = B^i$ $A^i \subseteq B^i$
- $A^i = B^i \cup C^i$ $A^i = B^i \cap C^i$ $A^i = B^i \setminus C^i$

for $0 \leq i \leq n$, and where A^i denotes a variable of sort i.

A given formula φ of nLS is said to be *satisfiable* if there exists a nonempty set S and an assignment M (mapping variables of sort i into elements of $pow^{(i)}(S)$) which makes the formula φ true, when the symbols of nLS are interpreted according to their standard meaning.

Since set variables of different sorts can interact by means of the membership predicate only, it is clear that a satisfiability test for formulae of nLS can be obtained in a straightforward manner from the satisfiability test for $2LS$.

[1]Some syntactical restrictions are placed on the sorts of X, Y, Z to meet the intuitive definition of the cartesian product of two sets.

A class of more expressive formulae is obtained by also allowing in the language nLS atoms of the following types

$$\begin{array}{ll} (\{\cdot\}) & A^i = \{B^{i-1}\} \\ (pow) & A^i = pow(B^{i-1}) \\ (Un) & B^{i-1} = Un(A^i) \end{array}$$

for $i \leq n$.

We denote such a class of formulae by $nLSSPU$.

In order to show that $nLSSPU$ has a solvable satisfiability problem we will reason as follows.

Given a satisfiable formula φ of $nLSSPU$ and a model M for φ, we define the *support* of M as the set $\mathcal{U}$ defined by

$$\mathcal{U} = \bigcup_{i=1,\dots,n} \bigcup_{A^i \in \mathcal{V}^i} Un^{(i)}(MA^i)\,, \tag{1}$$

where $\mathcal{V}^i$ denotes the collection of set variables of sort i occurring in φ, $0 \leq i \leq n$.

Therefore, for any $1 \leq i \leq n$ and for any $A^i \in \mathcal{V}^i$, we have that

$$MA^i \subseteq pow^{i-1}(\mathcal{U}). \tag{2}$$

We will exhibit a normalization procedure which allows to transform the given model M into another model M^* of φ whose support has a cardinality that can be a priori *bounded*, and which depends solely on the size of the formula φ. This clearly will imply the solvability of the satisfiability problem for $nLSSPU$.

For sake of clarity, we will first prove that the satisfiability problem for the class $3LSSPU$ is solvable. Subsequentely, we will extend such a result to the general case.

3 The class of formulae $3LSSPU$

In order to simplify our notation, we will use lower case letters $\dots, x, y, z$ to denote individual variables (variables of sort 0), capital letters $\dots, X, Y, Z$ to denote set variables (variables of sort 1), and capital letters $A, B, C, \dots$ to denote set of sets variables (variables of sort 2).

By using disjunctive normal form and elementary reductions, it is possible to reduce the satisfiability problem for the theory $3LSSPU$ to the same problem for *normalized conjunctions (n.c.)* of $3LSSPU$. These are conjunctions of literals of the following types

- $X = \emptyset$, $A = \emptyset$, $X = Y \cup Z$, $A = B \cup C$, $X = Y \setminus Z$, $A = B \setminus C$
- $X = \{x\}$, $A = \{X\}$, $A = pow(X)$, $X = Un(A)$.

Let φ be a n.c. of $3LSSPU$ and let $\mathcal{V}^i$ be the collection of variables of sort $i = 0, 1, 2$ occurring in φ. We say that φ is a *closed normalized conjunction (c.n.c.)* if the following conditions hold:

(1) the literal $A_0 = \{\emptyset\}$ occurs in φ;

(2) for all X in $\mathcal{V}^1$ there is a literal $A = pow(X)$ in φ, for some variable A in $\mathcal{V}^2$;

(3) for all A in $\mathcal{V}^2$ there is a literal $X = Un(A)$ in φ, for some variable X in $\mathcal{V}^1$.

A c.n.c. φ^* can always be obtained from a n.c. φ in the following way:

(step 0) $\varphi^* := \varphi$;

(step 1) for all A of type 2 in φ introduce a new variable X_A of type 1 and put in φ^* the conjunct $X_A = Un(A)$;

(step 2) for all X of type 1 in φ^* introduce a new variable A_X of type 2 and put in φ^* the conjunct $A_X = pow(X) \wedge X = Un(A_X)$

(step 3) introduce a new variable of type 2 A_0 and put $A_0 = \{\emptyset\} \wedge \emptyset = Un(A_0) \wedge A_0 = pow(\emptyset)$.

Therefore from now on, without loss of generality, we will focus our attention on the satisfiability problem for closed normalized conjunctions of $3LLSPU$.

The following lemma, which will be proved in the following section, guarantees the existence of certain models, that we call *normal models.*

LEMMA 3.1 *Let φ be a satisfiable c.n.c. of $3LSSPU$ and let $\mathcal{V}^i$ be the collection of variables of sort $i = 0, 1, 2$ occurring in φ. Then there exists a model M of φ such that denoted by $\sigma_1, \ldots, \sigma_k$ the disjoint Venn regions of the sets MA, for A in $\mathcal{V}^2$, for all $h = 1 \ldots k$ there exist $E_h \subseteq \sigma_h$ such that*

(a) $Un(E_h) \subseteq Un(\sigma_h)$;

(b) $\{MX : X \text{ in } \mathcal{V}^1\} \subseteq \bigcup_{h=1}^{k} E_h$;

(c) $|\bigcup_{h=1}^{k} E_h| \leq 3n \cdot 2^{2m}$,

where $n = |\mathcal{V}^1|$ *and* $m = |\mathcal{V}^2|$. $\square$

Let φ be a cnc of $3LSSPU$ and let M be a normal model for φ, with E_h, $h = 1 \ldots k$, as in Lemma 3.1.

Let $n = |\mathcal{V}^1|$, $m = |\mathcal{V}^2|$, $\mathcal{E} = \bigcup_{h=1}^{k} E_h$, and let $\mathcal{U}$ be the support of M.

The following procedure produces a subset $\mathcal{U}^*$ of $\mathcal{U}$ which will be used as support for a bounded model of φ.

Procedure 1
Input: collection $\mathcal{E}$;
Output: set $\mathcal{U}^*$;

$\mathcal{U}^* := \{Ma : a \in \mathcal{V}^0\}$;
for any pair $I_1, I_2 \in \mathcal{E}$ do
  if $I_2 \not\subseteq I_1$ then
    pick w in $I_2 \setminus I_1$;
    $\mathcal{U}^* := \mathcal{U}^* \cup \{w\}$;
  end if;
end for;
return $\mathcal{U}^*$;

end Procedure.

It is easily seen that since $|\mathcal{E}| \leq 3n \cdot 2^{2m}$, then $|\mathcal{U}^*| \leq 9n^2 \cdot 2^{4m}$.

Now we define an assignment M^* having as support the set $\mathcal{U}^*$. We put:

(t1) $M^*x = Mx$, for x in $\mathcal{V}^0$;

(t2) $M^*X = MX \cap \mathcal{U}^*$, for X in $\mathcal{V}^1$;

(t3) $M^*A = (MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^*$, for A in $\mathcal{V}^2$;

where

$$\begin{aligned} \mathcal{U}_2^* &= pow(\mathcal{U}^*) \setminus \{I \cap \mathcal{U}^* : I \in \mathcal{E}\} \\ \mathcal{E}_A^* &= \{I \cap \mathcal{U}^* : I \in MA \cap \mathcal{E}\}. \end{aligned}$$

It only remains to be proven that the assignment M^* as defined above is a model for φ. To this end we will use the following two elementary lemmas.

LEMMA 3.2 *For any sets s, t, u and v*

(a) $(s \setminus t) \cap u = (s \cap u) \setminus (t \cap u)$;

(b) *if* $(s \cup t) \cap (u \cup v) = \emptyset$ *then* $(s \setminus t) \cup (u \setminus v) = (s \cup u) \setminus (t \cup v)$. $\square$

LEMMA 3.3 *(a) If* $MA = MB \cup MC$ *then* $\mathcal{E}_A^* = \mathcal{E}_B^* \cup \mathcal{E}_C^*$;

(b) if $MA = MB \setminus MC$ *then* $\mathcal{E}_A^* = \mathcal{E}_B^* \setminus \mathcal{E}_C^*$;

(c) $\mathcal{U}_2^* \cap \mathcal{E}_A^* = \emptyset$, *for all A in* $\mathcal{V}^2$.

Proof. (a) Let $MA = MB \cup MC$. Then

$$\begin{aligned}
\mathcal{E}_A^* &= \{I \cap \mathcal{U}^* : I \in MA \cap \mathcal{E}\} = \{I \cap \mathcal{U}^* : I \in (MB \cup MC) \cap \mathcal{E}\} \\
&= \{I \cap \mathcal{U}^* : I \in MB \cap \mathcal{E}\} \cup \{I \cap \mathcal{U}^* : I \in MC \cap \mathcal{E}\} \\
&= \mathcal{E}_B^* \cup \mathcal{E}_C^* .
\end{aligned}$$

(b) Let $MA = MB \setminus MC$. If $J \in \mathcal{E}_A^*$ then $J = I \cap \mathcal{U}^*$ for some $I \in MA \cap \mathcal{E}$. Clearly $I \in MB \cap \mathcal{E}$, so that $J \in \mathcal{E}_B^*$. Moreover $I \notin MC$. If $J \in \mathcal{E}_C^*$, then $J = I' \cap \mathcal{U}^*$ for some $I' \in MC \cap \mathcal{E}$. But then by the strong version of M.G. we would have $I \cap \mathcal{U}^* \neq I' \cap \mathcal{U}^*$ with $I \neq I'$, a contradiction. Hence $J \notin \mathcal{E}_C^*$ which implies $\mathcal{E}_A^* \subseteq \mathcal{E}_B^* \setminus \mathcal{E}_C^*$.

Conversely, if $J \in \mathcal{E}_B^* \setminus \mathcal{E}_C^*$, then $J = I \cap \mathcal{U}^*$ for some $I \in MB \cap \mathcal{E}$. Obviously $I \notin MC$, $I \in (MB \setminus MC) \cap \mathcal{E} = MA \cap \mathcal{E}$, implying $J \in \mathcal{E}_A^*$ and in turn $\mathcal{E}_B^* \setminus \mathcal{E}_C^* \subseteq \mathcal{E}_A^*$. Thus in conclusion we have $\mathcal{E}_A^* = \mathcal{E}_B^* \setminus \mathcal{E}_C^*$.

Finally, property (c) follows immediately form the definition of $\mathcal{U}_2^*$. $\blacksquare$

Next we show that the assignment M^* defined by (t1)-(t3) above is a model for φ. We proceed by cases.

(case 1) Conjuncts of type $(\{\cdot\})$.

(1.1) Suppose that $X = \{a\}$ is in φ.

Then $M^*X = MX \cap \mathcal{U}^* = \{Ma\} \cap \mathcal{U}^* = \{M^*a\} \cap \mathcal{U}^* = \{M^*a\}$.

(1.2) Suppose that $A = \{X\}$ is in φ.

Since $\{MX\} \cap \mathcal{U}_2^* \subseteq \{M^*X\}$ and $\mathcal{E}_A^* = \{M^*X\}$, it follows that $M^*A = (MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^* = \{M^*X\}$.

(case 2) Conjuncts of type ($\cup$).

(2.1) Suppose $X = Y \cup Z$ is in φ.

Plainly, $M^*X = MX \cap \mathcal{U}^* = (MY \cup MZ) \cap \mathcal{U}^* = (MY \cap \mathcal{U}^*) \cup (MZ \cap \mathcal{U}^*) = M^*Y \cup M^*Z$.

(2.2) Suppose that $A = B \cup C$ is in φ.

Then by Lemma 3.3(a)

$$\begin{aligned} M^*A &= (MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^* = [(MB \cup MC) \cap \mathcal{U}_2^*] \cup (\mathcal{E}_B^* \cup \mathcal{E}_C^*) \\ &= [(MB \cap \mathcal{U}_2^*) \cup \mathcal{E}_B^*] \cup [(MC \cap \mathcal{U}_2^*) \cup \mathcal{E}_C^*] = M^*B \cup M^*C \end{aligned}$$

(case 3) Conjuncts of type ($\setminus$).

(3.1) Suppose that $X = Y \setminus Z$ is in φ.

Then by Lemma 3.2(a), $M^*X = MX \cap \mathcal{U}^* = (MY \setminus MZ) \cap \mathcal{U}^* = (MY \cap \mathcal{U}^*) \setminus (MZ \cap \mathcal{U}^*) = M^*Y \setminus M^*Z$.

(3.2) Suppose that $A = B \setminus C$ is in φ.

Then by Lemmas 3.2 and 3.3(b),(c) we have

$$\begin{aligned} M^*A &= (MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^* = [(MB \setminus MC) \cap \mathcal{U}_2^*] \cup (\mathcal{E}_B^* \setminus \mathcal{E}_C^*) \\ &= [(MB \cap \mathcal{U}_2^*) \cup \mathcal{E}_B^*] \setminus [(MC \cap \mathcal{U}_2^*) \cup \mathcal{E}_C^*] = M^*B \setminus M^*C \end{aligned}$$

(case 4) Conjuncts of type (pow).

Suppose that $A = pow(X)$ is in φ.

We need to show that $(MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^* = pow(MX \cap \mathcal{U}^*)$. So let $J \in (MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^*$. If $J \in MA \cap \mathcal{U}_2^*$ then $J \in pow(MX) \cap pow(\mathcal{U}^*) = pow(MX \cap \mathcal{U}^*)$. On the other hand if $J \in \mathcal{E}_A^*$, then $J = I \cap \mathcal{U}^*$ for some $I \in MA \cap \mathcal{E}$. But then $I \subseteq MX$, so that $J \subseteq MX \cap \mathcal{U}^*$, which yields $J \in pow(MX \cap \mathcal{U}^*)$. Summing up, we have proved that $(MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^* \subseteq pow(MX \cap \mathcal{U}^*)$.

Conversely, let $J \in pow(MX \cap \mathcal{U}^*)$. Assume that $J \notin MA \cap \mathcal{U}_2^*$. Since $J \subseteq MX$, we have $J \in MA$. In addition, $J \subseteq \mathcal{U}^*$ so that $J \in pow(\mathcal{U}^*)$. Thus we must have $J \in \{I \cap \mathcal{U}^* : I \in \mathcal{E}\}$. Let $J = I \cap \mathcal{U}^*$ for some $I \in \mathcal{E}$. Observe that $I \in MA$, because otherwise $I \not\subseteq MX$ which by the strong version of the M.G. would imply $J = I \cap \mathcal{U}^* \not\subseteq MX \cap \mathcal{U}^*$, i.e. $J \notin pow(MX \cap \mathcal{U}^*)$. Hence $J \in \mathcal{E}_A^*$ which in conclusion implies $pow(MX \cap \mathcal{U}^*) \subseteq (MA \cap \mathcal{U}_2^*) \cup \mathcal{E}_A^*$.

We can then conclude that $M^*A = pow(M^*X)$.

(case 5) Conjuncts of type (Un).

Suppose that $X = Un(A)$ is in φ.

Since M is a normal model (cf. Lemma 3.1) there exist sets $I_1, \ldots, I_h$ in $MA \cap \mathcal{E}$ such that $MX = \bigcup_{j=1,\ldots,h} I_j$. This implies that $M^*X = (\bigcup_{j=1,\ldots,h} I_j) \cap \mathcal{U}^* = \bigcup_{j=1,\ldots,h}(I_j \cap \mathcal{U}^*)$. Since $I_j \cap \mathcal{U}^* \in M^*A$, for $j = 1, \ldots, h$, we have $M^*X \subseteq Un(M^*A)$.

On the other hand, if $u \in Un(M^*A)$ it follows that $u \in \mathcal{U}^*$ and $u \in J$, for some $J \in M^*A$. Two cases are possible.

(5.1) $J \in MA \cap \mathcal{U}_2^*$.

In this case $J \subseteq MX$, so that $u \in J \cap \mathcal{U}^* \subseteq MX \cap \mathcal{U}^* = M^*X$.

(5.2) $J \in \mathcal{E}_A^*$.

In this case $J = I \cap \mathcal{U}^*$ for some $I \in MA \cap \mathcal{E}$. Thus $I \subseteq MX$ and $u \in J = I \cap \mathcal{U}^* \subseteq MX \cap \mathcal{U}^* = M^*X$, and again $u \in M^*X$.

It follows that $Un(M^*A) \subseteq M^*X$ which together with $M^*X \subseteq Un(M^*A)$ proved earlier gives $M^*X = Un(M^*A)$.

Then we have the following lemma.

LEMMA 3.4 *A c.n.c. of $3LSSPU$ with n variables of sort 1 and m variables of sort 2 is satisfiable if and only if it has a model with a support whose size is bounded by $9n^2 \cdot 2^{4m}$.* ■

We can then conclude with the following theorem.

THEOREM 3.1 *The class of formulae $3LSSPU$ has a solvable satisfiability problem.* ■

4 Proof of Lemma 3.1

Let φ be a satisfiable c.n.c. of $3LSSPU$. Let M be a model of φ and let $\mathcal{U}$ be its support. As above for $i = 0, 1, 2$, denote by $\mathcal{V}^i$ the collection of variables of type i occurring in φ and let $n = |\mathcal{V}^1|$.

Let $\sigma_1, \ldots, \sigma_k$ be the Venn regions of the sets MA for $A \in \mathcal{V}^2$. Thus to each A in $\mathcal{V}^2$ we can associate a set $H_A \subseteq \{1, \ldots, k\}$ such that

$$MA = \bigcup_{h \in H_A} \sigma_h \tag{3}$$

Since φ is closed for all $1 \leq h \leq k$ we have

(P1) if $\alpha \subseteq \sigma_h$ then $Un(\alpha) \in \bigcup_{\ell=1}^{k} \sigma_\ell$;

(P2) if there exists $X \in \mathcal{V}^1$ such that $MX \in \sigma_h$ then $Un(\sigma_h) = MX$.

Let us consider the following two procedures whose effect is to modify some of the Venn regions $\sigma_1, \ldots, \sigma_k$.

Procedure $TRANS_1(h, \ell, I)$ **Input:** $h, \ell \in \{1, \ldots, k\}$; $I \in pow^{(1)}(\mathcal{U})$. **Output:** modified σ_h, σ_ℓ. $\sigma_h := \sigma_h \setminus \{I\}$; $\sigma_\ell := \sigma_\ell \cup \{I\}$; return; **end Procedure.**	**Procedure** $TRANS_2(h, \ell, \alpha)$ **Input:** $h, \ell \in \{1, \ldots, k\}$; $\alpha \subseteq pow^{(2)}(\mathcal{U})$. **Output:** modified σ_h, σ_ℓ. $\sigma_h := (\sigma_h \setminus \{Un(\alpha)\}) \cup \alpha$; $\sigma_\ell := (\sigma_\ell \setminus \alpha) \cup \{Un(\alpha)\}$; return; **end Procedure.**

By using properties (P1) and (P2) above, we will now show that if for some $h, \ell \in \{1, \ldots, k\}$, $I \in pow^{(1)}(\mathcal{U})$ we have

(a) $I \in \sigma_h$;

(b) $I \neq MX$, for all $X \in \mathcal{V}^1$;

(c) $Un(\sigma_h \setminus \{I\}) = Un(\sigma_h)$;

(d) $I \subseteq Un(\sigma_\ell) \subseteq Un(\sigma_h)$;

(e) σ_ℓ is not a singleton of type $\{MX\}$, for any $X \in \mathcal{V}^1$;

then after the execution of $TRANS_1(h, \ell, I)$ (and the subsequent modification of the sets σ_i) the assignment M' defined by

(i) $M'x = Mx$, for x in $\mathcal{V}^0$;

(ii) $M'X = MX$, for X in $\mathcal{V}^1$;

(iii) $M'A = \bigcup_{i \in H_A} \sigma_i$, for A in $\mathcal{V}^2$;

is still a model for φ.

Likewise, if for some $h, \ell \in \{1, \ldots, k\}$ and $\alpha \subseteq pow^{(2)}(\mathcal{U})$ we have

(f) $\alpha \subseteq \sigma_\ell$;

(g) $Un(\alpha) \subseteq \sigma_h$;

(h) $Un(\alpha) \notin \{MX : X \text{ in } \mathcal{V}^1\}$;

then after the execution of $TRANS_2(h, \ell, \alpha)$ (and the subsequent modification of the sets σ_i) the assignment M' formally defined as above is a model for φ.

Indeed, in both cases we have that

- the sets σ_h remain pairwise disjoint then M' models the conjuncts of type $(\cup), (\setminus)$;

- because of (d) and (h) neither σ_h nor σ_ℓ can be singletons of type $\{MX\}$ for some $X \in \mathcal{V}^1$. Therefore, conjuncts of type $(\{\cdot\})$ are well modeled too;

- $Un(\sigma_h)$ and $Un(\sigma_\ell)$ remain unchanged, therefore conjuncts of type (Un) are well modeled.

Finally, notice that $\sigma_h \cup \sigma_\ell$ does not change. This allows us to claim that also conjuncts of type (pow) are well modeled by M'. Let us consider separately the two cases.

(a) Suppose first that the applied procedure is $\mathcal{TRANS}_1$. If $h \in H_A$ for some $A = pow(X)$ conjunct of φ, then $I \subseteq Un(\sigma_h) \subseteq MX$, therefore $I \in MA$ and so $\ell \in H_A$. Conversely, suppose that $\ell \in H_A$ for some $A = pow(X)$ conjunct of φ, then $I \subseteq Un(\sigma_\ell) \subseteq MX$. On the other hand, by (c) $Un(\sigma_h) \subseteq MX$ therefore $\sigma_h \subseteq MA$ and so $h \in H_A$.

(b) Suppose now that the applied procedure is $\mathcal{TRANS}_2$. If $h \in H_A$ for some $A = pow(X)$ conjunct of φ, then $Un(\alpha) \subset MX$ so $\sigma_\ell \subseteq MA$, i.e. $\ell \in H_A$. Analogously, if $\ell \in H_A$ and $Un(\alpha) \in MA$ then all elements of α are subsets of MX i.e. $\alpha \subset MA$ and then $h \in H_A$.

Now we are ready to exhibit a procedure that normalizes models in the sense of Lemma 3.1.

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Procedure Normalize;
Input: collection σ_1, ..., σ_k;
Output: modified σ_1, ..., σ_k; the set ℰ;

    ℰ := {MX : X ∈ 𝒱^1};
    for ℓ := 1 to k such that σ_ℓ is infinite do
        I := Un(σ_ℓ);
        pick distinct u, v, w, z ∈ I such that
            (A1) there exists h ∈ {1, ..., k} such that I \ {a} ∈ σ_h, for a = u, v, w, z;
            (A2) I \ {a} ∉ ℰ, for a = u, v, w, z;
        TRANS_1(h, ℓ, I \ {u});
        TRANS_1(h, ℓ, I \ {v});
        ℰ := ℰ ∪ {I \ {u}, I \ {v}};
    end for;
    for ℓ := 1 to k such that σ_ℓ is finite do
        while there exists α ⊆ σ_ℓ such that Un(α) ∉ ℰ ∪ σ_ℓ do
            pick h ∈ {1, ..., k} such that Un(α) ∈ σ_h;
            TRANS_2(h, ℓ, α);
        end while;
        pick the minimum α ⊆ σ_ℓ (w.r.t. cardinality ) such that
            • Un(α) = Un(σ_ℓ);
        ℰ := ℰ ∪ α;
    end for;
    return ℰ;

end Procedure.
```

Let M^+ be the assignment defined as in (i)-(iii) above after the execution of the first for loop of the procedure *Normalize* applied on the given model M of φ. From the previous discussion, it follows immediately that M^+ is a model for φ. Moroever, M^+ verifies the following property:

Finite Property (FP): for any Venn region $\sigma \subseteq M^+A$ for some $A \in \mathcal{V}^2$, there exists $\alpha \subseteq \sigma$ such that

(FP1) α is finite and

(FP2) $Un(\alpha) = Un(\sigma)$.

Notice that given h, if it is possible to find $u, v, w, z \in I = Un(\sigma_h)$ such that (A1) and (A2) are verified then

$$\begin{array}{rcl} Un(\sigma_h) & \subseteq & Un(\sigma_\ell) \\ Un(\sigma_{h'}) & = & Un(\sigma_\ell \setminus \{I \setminus \{u\}, I \setminus \{v\}\} \\ Un(\sigma_h) & = & Un(\sigma_h \cup \{I \setminus \{u\}, I \setminus \{v\}\} \\ Un(\sigma_h) & = & I \setminus \{u\} \cup I \setminus \{v\} \end{array}$$

Therefore, we only have to prove that for any infinite σ_h it is always possible to find four distinct elements of its union verifying (A1) and (A2).

So, let σ_h be an infinite Venn region. Since $\mathcal{E}$ is finite there clearly exist infinitely many elements of type $Un(\sigma_h) \setminus \{a\}$ for $a \in Un(\sigma_h)$, that do not belong to $\mathcal{E}$. Moreover, since each of these elements belongs to one Venn region, there clearly exists one Venn region σ_ℓ which contains at least four of them.

For $i = 1, \ldots, k$, put $E_i = \sigma_i \cap \mathcal{E}$, where the values of the sets σ_i are meant after the execution of the procedure *Normalize*. Then properties (a) and (b) of Lemma 3.1 follow easily. Concerning property (c), we reason as follows. Let now α_ℓ be the minimum subset of σ_ℓ chosen in the second for-loop of the procedure *Normalize*, for each $\ell = 1, \ldots, k$. Moreover, let $\mathcal{E}^\ell$ be the set-value of the variable $\mathcal{E}$ just after the the execution of step ℓ of the same for-loop.

By hypothesys we have that for all $\beta \subseteq \alpha_\ell$ $Un(\beta) \in \mathcal{E} \cup \sigma_\ell$ Suppose that there exist $\beta_1, \beta_2 \subseteq \alpha_h$, $\beta_1 \neq \beta_2$, such that $Un(\beta_1) = Un(\beta_2)$ and suppose by contradiction that $\beta_1 \not\subseteq \beta_2$.

So, let $I \in \beta_1 \setminus \beta_2$, and let us prove that $Un(\alpha_h \setminus \{I\}) = Un(\alpha_h)$, which would contradict the hypothesys that α_h is minimal.

Let $w \in I$, since $Un(\beta_2) = Un(\beta_1)$ there exists $I' \in \beta_2$ such that $w \in I'$. Therefore $w \in Un(\alpha_h \setminus \{I\})$.

Analogously, it can be proven that $\beta_2 \subseteq \beta_1$.

The elements of $pow(\alpha_h)$ can then injectively be mapped into the elements of $\mathcal{E}^{\ell-1} \cup \sigma_\ell$. In particular, in view of the minimality of α its subsets with more than one element can be injectively mapped into $\mathcal{E}^{\ell-1}$. Therefore $2^{|\alpha|} - |\alpha| \leq \mathcal{E}^{h-1}$ which implies that $|\alpha| \leq 2 \log \mathcal{E}^{h-1}$.

It turns out that $|\mathcal{E}|$ is bounded by the solution of the following recurrence equation:

$$
\begin{aligned}
T(0) &= 3n \\
T(\ell) &= T(\ell - 1) + 2 \log T(\ell - 1)
\end{aligned}
$$

By induction, we have easily $T(\ell) \leq 3n(\ell + 1)^2$, for $\ell \geq 0$. Thus, since $\ell \leq 2^m - 1$, we have $|\mathcal{E}| \leq 3n \cdot 2^{2m}$, concluding the proof that M' is a normal model and in turn the proof of Lemma 3.1.

5 The class 4LSSP

Before proving that the results of the previous section can be extended to the class of formulae where $n > 3$ different sorts of variables are allowed, i.e. the class $nLSSPU$, it is useful to investigate the satisfiability problem for the class of formulae $4LSSP$, i.e. the class where four different sorts of variables occur and where only the powerset and singleton operators are present. Such a class of formulae has a solvable satisfiability problem in view of the results in [Can91]. However, in this special case, we wish to introduce a new technique that can be used and extended when also occurrences of the union operators are allowed.

Let us now define more specifically the class of formulae $4LSSP$ as the propositional closure of atoms of types

$(\emptyset)$ $X = \emptyset,\ \ A = \emptyset\ \ S = \emptyset$

$(\in)$ $a \in X,\ \ X \in A\ \ A \in S$

$(=)$ $a = b,\ \ X = Y,\ \ A = B\ \ S = T$

$(\subseteq)$ $X \subseteq Y,\ \ A \subseteq B\ \ S \subseteq T$

$(\cup)$ $X = Y \cup Z,\ \ A = B \cup C\ \ S = T \cup U$

$(\cap)$ $X = Y \cap Z,\ \ A = B \cap C\ \ S = T \cap U$

$(\setminus)$ $X = Y \setminus Z,\ \ A = B \setminus C\ \ S = T \setminus U$

$(\{\cdot\})$ $X = \{a\},\ \ A = \{X\}\ \ S = \{A\}$

(pow) $A = pow(X)\ \ S = pow(A)$

where lower case letters $a, b, c, \ldots$ denote element variables (type 0), capital letters X,Y,Z denote set variables (type 1), capital letters A, B, C denote sets of sets variables (type 2) and capital letters S, T, U denote sets of sets of sets variables (type 3).

As usual, we will restrict ourselves to the satisfiability problem for normalized conjunctions defined as conjunctions of literals of type

$(\emptyset)$ $X = \emptyset,\ \ A = \emptyset\ \ S = \emptyset$

($\cup$) $X = Y \cup Z,\ \ A = B \cup C\ \ S = T \cup U$

($\setminus$) $X = Y \setminus Z,\ \ A = B \setminus C\ \ S = T \setminus U$

($\{\cdot\}$) $X = \{a\},\ \ A = \{X\}\ \ S = \{A\}$

(pow) $A = pow(X)\ \ S = pow(A)$

We will also suppose that such conjunctions are *closed,* meaning in this case that for every variable of type i with $0 < i < 3$ there exists a variable of type $i+1$ that represents the powerset of the given variable. As usual our goal is to show that given a satisfiable formula φ of $4LSSP$ and a model M with support $\mathcal{U}$, there exists a model M^* with support $\mathcal{U}^*$ whose cardinality is a priori bounded in the size of φ.

By recalling the results in [Can91] (or by using $\mathcal{MOD}_1$ of the previous section) we can claim that $\mathcal{U}$ is finite.

Let now $A_1, \ldots, A_n$ be the variables of type 2 occurring in φ and $S_1, \ldots, S_m$ be the variables of type 3 occurring in φ. Since φ is closed for any A_j there exists S_i such that $S_i = pow(A_j)$ is in φ and therefore $MS_i = pow(MA_j)$.

Let $\sigma_1, \ldots, \sigma_l$ be the nonempty and disjoint Venn regions of the sets MS_i for $i = 1, \ldots, m$. For each region σ let I_σ be an element of σ which is maximal in σ with respect to the inclusion. Denote by $\mathcal{E}$ the collection of these elements I_σ.

Let also $H \subseteq \{1, \ldots, n\}$ and let

$$\mathcal{S}_H = \bigcap_{h \in H} pow(MA_h) \setminus \bigcup_{h \notin H} pow(MA_h) \tag{4}$$

and

$$I_H = \bigcap_{h \in H} MA_h. \tag{5}$$

For all $H \subseteq \{1, \ldots, k\}$ we have that

(R1) $\mathcal{S}_H = \sigma_{1,H} \cup \sigma_{2,H} \cdots \cup \sigma_{j_H,H}$;

(R2) if for some $I \subseteq I_H$ it is the case that $I \notin \mathcal{S}_H$ then for all $I' \subseteq I$ it must be the case that $I' \notin \mathcal{S}_H$.

In particular, if $\mathcal{S}_H \neq \emptyset$ then $I_H \in \mathcal{S}_H$;

(R3) there exists at most one element of type MA_j, for $1 \leq j \leq n$, in $\mathcal{S}_H$ and one such an element must be exactly I_H;

(R4) if $\mathcal{S}_H \neq \emptyset$ then $I_H \in \mathcal{E}$;

(R5) if $\mathcal{S}_H \neq \emptyset$, $I_H = MA_i$ for some $1 \leq i \leq n$ and A_i is a singleton variable, then there exists $1 \leq h \leq j_H$ (see R1) such that $\sigma_h = \{MA_i\} = \{I_H\}$.

The above properties are easy to prove. Their generalization to the case $nLSSPU$ will be treated in the next section.

To each of the sets I in $\mathcal{E}$ we associate a newly introduced variable A_I of type 2 and we extend M over these new variables by putting $MA_I = I$.

Let $\mathcal{V}^1$ be the collection of variables of type 1 in φ and let $\mathcal{V}^2$ be the collection of variables of type 2 in φ plus the above introduced variables.

Let us now consider the following formula of $3LSSP$, φ^* defined as

$$\varphi^- \wedge \psi$$

where

(a) φ^- is the subconjunction of φ not involving variables of type 3;

(b) ψ is a formula that explicits all relations of type $\cup, \setminus, \{\cdot\}$ and $\in, pow$ among the elements of $\mathcal{E}$ and all the other sets of type 2 and 1, that are associated by M to the variables of type 2 and 1; i.e.

  (1) for any A_1, A_2 in $\mathcal{V}^2$, if $MA_1 \subseteq MA_2$ then $A_1 \subseteq A_2$ is in ψ;

  (2) for any A_1, A_2, A_3 in $\mathcal{V}^2$, if $MA_1 = MA_2 \cup MA_3$ [resp. $MA_1 = MA_2 \cap MA_3$, $MA_1 = MA_2 \setminus MA_3$] then $A_1 = A_2 \cup A_3$ [resp. $A_1 = A_2 \cap A_3$, $A_1 = A_2 \setminus A_3$] is in ψ;

  (3) for any $X \in \mathcal{V}^1$ and any $A \in \mathcal{V}^2$, if $MA = \{MX\}$ [resp. $MX \in MA$] then $A = \{X\}$ [resp. $X \in A$] is in ψ.

The variables occurring in ψ are then the new variables above introduced and the variables occurring in φ of type $n-2$ and $n-1$.

Since φ^* is also in particular a formula of $3LSSPU$, we can conclude that it has a model M^* whose support is a priori bounded in its size and therefore a priori bounded in the size of φ.

It can easily be checked that by putting

$$M^*S = \{M^*A : MA \in MS\},$$

we obtain an assignment on the variables of type 3 which satisfies all conjuncts of φ but the conjuncts of type (pow).

We now proceed as follows.

As in (4), let $H \subseteq \{1, \ldots, n\}$ and let

$$\begin{aligned} \mathcal{S}^*_H &= \bigcap_{h\in H} pow(M^*A_h) \setminus \bigcup_{h\notin H} pow(M^*_h) \\ \mathcal{T}_H &= \bigcap_{h\in H} MS_h \setminus \bigcup_{h\notin H} MS_h \end{aligned}$$

where for any h S_h is the variable of type 3 related to the variable A_h of type 2 by the conjunct of φ, $S_h = pow(A_h)$. Therefore,

(a) $MS_h = pow(MA_h)$

(b) $\mathcal{T}_H = \mathcal{S}_H$ for any $H \subseteq \{1, \ldots, k\}$

(c) for all $H \subseteq \{1, \ldots, k\}$, $\mathcal{S}^*_H = \emptyset$ if and only if $\mathcal{T}_H = \emptyset$.

Put now,

$$\sigma^* = \{M^*A_I : I \in \sigma\} \tag{6}$$

Call a Venn region σ *singleton* if and only if $\sigma = \{MA\}$ for some A singleton variable occurring in φ.

Let also

$$\mathcal{R}^*_H = \mathcal{S}^*_H \setminus \{M^*A : A \text{ occurring in } \varphi^*\}. \tag{7}$$

Let $H \subseteq \{1, \ldots, n\}$ such that $\mathcal{R}^*_H \neq \emptyset$. Our goal is to put the elements of $\mathcal{R}^*_H$ in the sets σ^* in such a way that

(1) the sets σ^* remain pairwise disjoint;

(2) if σ is singleton so is σ^*

If we can accomplish this we can certainly claim that the assignment M^* extended over the variables of φ of type 3 in the following way

$$M^*S = \bigcup_{\sigma \subseteq MS} \sigma^* \tag{8}$$

is a model for φ.

Two cases are now possible

(case 1) For any $\sigma \subseteq \mathcal{S}_H$ it is the case that σ is a singleton.

In this case, in view of (R3) above, we have that $\mathcal{S}_H = \{MA\}$ and so $\mathcal{S}_H^* = \{M^*A\}$ which implies that $\mathcal{R}_H^*$ is empty.

(case 2) There exists $\sigma \subseteq \mathcal{S}_H$ which is not singleton. Therefore, we put

$$\sigma^* := \sigma^* \cup \mathcal{R}_H^*.$$

This model-extension technique above described, can easily be applied with a recursive argument to the case in which there are n different types of variables, i.e. for the class of formulae $nLSSP$.

In the next section, we will show how it can be extended in the general case in which also the general union operator is allowed.

6 Extension to n-level syllogistic

The result of the previous section can be extended to classes of formulae where $n > 3$ different sorts of variables are allowed, i.e. to the class $nLSSPU$.

As before we can restrict ourselves to the satisfiability problem for *closed normalized conjunctions,* i.e. conjunctions of literals of type

($\emptyset$) $A^i = \emptyset$

($\cup$) $A^i = B^i \cup C^i$

($\setminus$) $A^i = B^i \setminus C^i$

$(\{\cdot\})$ $A^i = \{B^{i-1}\}$

(pow) $A^i = pow(B^{i-1})$

(Un) $B^{i-1} = Un(A^i)$

for $1 \leq i \leq n$, where

- for any $2 \leq i \leq n-1$ and any A^i occurring in φ there exists B^{i-1} and C^{i+1} such that the conjuncts
$$B^{i-1} = Un(A^i) \wedge C^{i+1} = pow(A^i)$$
are in φ;

- for any variable A^n occurring in φ there exists B^{n-1} such that
$$B^{n-1} = Un(A^n)$$
is in φ;

- for any variable A^1 occurring in φ there exists B^2 such that
$$B^2 = pow(A^1).$$

In the following we will call *singleton* any variable B^{i-1} appearing in a conjunct of type $A^i = \{B^{i-1}\}$. Given a normalized conjunction φ of $nLSSSPU$ it is straightforward to see that we can always produce a closed normalized conjunction from it.

Let then φ be a closed normalized conjunction of $nLSSPU$. Let for $0 \leq i \leq n$, $\mathcal{V}^i$ be the collection of set variables of type i occurring in φ. Since φ is closed for any A^k occurring in φ, $k \leq n$ and for any $k = 1, \ldots, n$, there exist variables A^k of type k occurring in φ. Suppose that φ is satisfiable and let M be a model for φ. Let $\mathcal{U}$ be the *support* of M.

Suppose $n > 3$ and consider the closed normalized conjunction φ^n that contains all and only the conjuncts of φ of type:

$(\emptyset)$ $A^n = \emptyset$, $A^{n-1} = \emptyset$

$(\cup)$ $A^n = B^n \cup C^n$, $A^{n-1} = B^{n-1} \cup C^{n-1}$

($\setminus$) $A^n = B^n \setminus C^n,\ A^{n-1} = B^{n-1} \setminus C^{n-1}$

($\{\cdot\}$) $A^n = \{B^{n-1}\},\ A^{n-1} = \{B^{n-2}\}$

(pow) $A^n = pow(B^{n-1})$

(Un) $B^{n-1} = Un(A^n)$

Therefore, φ^n is a $3LSSPU$ formula satisfied by an assignment M over a support $\mathcal{H} = pow^{n-2}(\mathcal{U})$. If we apply $\mathcal{MOD}_1$ and $\mathcal{MOD}_2$ to the Venn regions $\sigma_1^n, \ldots, \sigma_{k_n}^n$ of the sets MA^n we can claim that starting from M we obtain an assignment $M^{n,+}$ such that:

(c1) $M^{n,+}$ models φ^n;

(c2) $M^{n,+}A^{n-1} = MA^{n-1}$, for each variable A^{n-1} of type $n-1$;

(c3) $M^{n,+}A^{n-2} = MA^{n-2}$, for each variable A^{n-2} of type $n-2$;

(c4) Lemma 3.1 holds for the Venn regions $\sigma_1^n, \ldots, \sigma_{k_n}^n$ of the sets MA^n, i.e. for all $1 \leq h \leq k_n$ there exist s_h elements of σ_h^n, $I_{h1}^{n-1}, \ldots, I_{hs_h}^{n-1}$, such that

(c4.1)

$$\bigcup_{t=1,\ldots,s_h} I_{ht}^{n-1} = Un(\sigma_h^n).$$

(c4.2) Moreover, if σ_h^n is infinite then $s_h \leq 2$.

(c4.3) Finally, denoted by $E_h^n = \{I_{h1}^{n-1}, \ldots, I_{hs_h}^{n-1}\}$ and by $\mathcal{E}^n = \bigcup_{h=1,\ldots,k} E_h^n$ we have that

$$\mathcal{E}^n \supseteq \{MB^{n-1} : B^{n-1} \text{ occurring in } \varphi\}$$

and

$$|\mathcal{E}^n| \leq 2^{2m_{n-1}} m_{n-1}$$

where m_{n-1} is the number of variables of type n-1 occurring in φ.

It is fairly simple to observe that the assignment M^+ defined as

$$\begin{aligned} M^+A^k &= MA^k \text{ for } k < n \\ M^+A^n &= M^{n,+}A^n \end{aligned}$$

over all variables occurring in φ is a model for φ. Notice in particular, that the support of M^+ defined as in (1) is equal to the support of M.

Moreover, let $A_1^{n-1}, \ldots, A_k^{n-1}$ be the set variables of type $n-1$. Since φ is closed for any $1 \leq h \leq k$ there exists B_h^n variable of type n such that $MB_h^n = pow(MA_h^{n-1})$. Let now $H \subseteq \{1, \ldots, k\}$ and let

$$\mathcal{S}_H = \bigcap_{h \in H} pow(MA_h^{n-1}) \setminus \bigcup_{h \notin H} pow(MA_h^{n-1}) \tag{9}$$

and

$$I_H = \bigcap_{h \in H} MA_h^{n-1}. \tag{10}$$

Suppose then that M is a model for φ that satisfies the condition (c4) above described. We have that, for all $H \subseteq \{1, \ldots, k\}$

(Rem 1) $\mathcal{S}_H = \sigma_{1,H}^n \cup \sigma_{2,H}^n \cdots \cup \sigma_{j_H,H}^n$;

(Rem 2) if for some $I \subseteq I_H$ it is the case that $I \notin \mathcal{S}_H$ then for all $I' \subseteq I$ it must be the case that $I' \notin \mathcal{S}_H$.

In particular, if $\mathcal{S}_H \neq \emptyset$ then $I_H \in \mathcal{S}_H$;

(Rem 3) there exists at most one element of type MA_i^{n-1}, for $1 \leq i \leq k$, in $\mathcal{S}_H$ and one such an element must be exactly I_H;

(Rem 4) if $\mathcal{S}_H \neq \emptyset$ then $I_H \in \mathcal{E}^n$;

(Rem 5) if $\mathcal{S}_H \neq \emptyset$, $I_H = MA_i^{n-1}$ for some $1 \leq i \leq k$ and A_i is a singleton variable, then there exists $1 \leq l \leq j_H$ (see Rem 1) such that $\sigma_l^n = \{MA_i^{n-1}\} = \{I_H\}$. Suppose, w.l.o.g. that $l = 1$. Moreover, if for all $1 < l \leq j_H$, $Un(\sigma_{l,H}^n) \subset I_H$ then $|I_H| = |MA_i^{n-1}| \leq l - 1 + |\{1, \ldots, k\} \setminus H|$ and for any $S^{n-2} \in I_H$, we have that $I_H \setminus \{S^{n-2}\} \in \mathcal{E}^n$.

Indeed, notice that for any $S^{n-2} \in I_H$, two cases are possible

(c1) either $I_H \setminus \{S^{n-2}\} \in \mathcal{S}_H$ or

(c2) $I_H \setminus \{S^{n-2}\} \subseteq MA_{h'}^{n-1}$ for some $h' \notin H$, i.e. $I_H \setminus \{S^{n-2}\} \in pow(MA_{h'}^{n-1})$.

Since, for all $1 < l \leq j_H$, $Un(\sigma_{l,H}^n) \subset I_H$ two distinct subsets of I_H of type $I_H \setminus \{S^{n-2}\}$ for $S^{n-2} \in I_H$ cannot belong both to a same Venn region contained in $\mathcal{S}_H$.

On the other hand, if two distinct subsets of I_H of type $I_H \setminus \{S^{n-2}\}$ for $S^{n-2} \in I_H$ are contained in a set $MA_{h'}^{n-1}$ for $h' \notin H$, then $I_H \subseteq MA_{h'}^{n-1}$ and $\mathcal{S}_H = \emptyset$.

So each Venn region $\sigma_{l,H}^n$ for $2 \leq l \leq j_H$ contains at most one element of type $I_H \setminus \{S^{n-2}\}$ for $S^{n-2} \in I_H$. Analogously, each set $pow(MA_{h'}^{n-1}$ for $h' \notin H$ contains at most one element of type $I_H \setminus \{S^{n-2}\}$ for $S^{n-2} \in I_H$.

This proves that $|I_H| = |MA_i^{n-1}| \leq l - 1 + |\{1, \ldots, k\} \setminus H|$.

Let now $S^{n-2} \in I_H$. If (case (c1)) $I_H \setminus \{S^{n-2}\} \in \mathcal{S}_H$ then $I_H \setminus \{S^{n-2}\} \in \sigma_{l,H}^n$, for some $2 \leq l \leq j_H$. So, $Un(\sigma_{l,H}^n) = I_H \setminus \{S^{n-2}\}$ and therefore $I_H \setminus \{S^{n-2}\}$ certainly belongs to $\mathcal{E}^n$.

If, on the other hand (case (c2)), $I_H \setminus \{S^{n-2}\} \in pow(MA_{h'}^{n-1})$ for some $h' \notin H$, let H' be the subsets of $\{1, \ldots, k\}$ which contains the indices of all the variables A^{n-1} such that $MA^{n-1} \supseteq I_H \setminus \{S^{n-2}\}$. Clearly, $i, h' \in H'$ and since $MA_{h'}^{n-1} \cap MA_i^{n-1} = I_H \setminus \{S^{n-2}\}$, we have $I_{H'} = I_H \setminus \{S^{n-2}\}$ therefore proving that there exists a Venn region σ^n whose union is $I_{H'}$ and to which $I_{H'}$ belongs. This assures us that $I_{H'} \in \mathcal{E}^n$.

We now proceed as follows. For any set I^{n-1} of type $n-1$ in $\mathcal{E}^n$, introduce a new variable $A_{I^{n-1}}^{n-1}$ of type $n-1$. Extend M over these new variables by putting $MA_{I^{n-1}}^{n-1} = I^{n-1}$.

Let us now consider the following formula of $(n-1)LSSPU$, φ^* defined as

$$\varphi^- \wedge \psi$$

where

(a) φ^- is the subconjunction of φ not involving variables of type n;

(b) ψ is a formula that explicits all relations of type $\cup, \setminus, \{\cdot\}$ and $\in, Un, pow$ among the elements of $\mathcal{E}^n$ and all the other sets of type $n-1$ and $n-2$. The variables occurring in ψ are then the new variables above introduced and the variables occurring in φ of type $n-2$ and $n-1$.

By applying a recursive argument, suppose that φ^* has an injective model M^* a priori bounded, i.e. with a support $\mathcal{U}^*$ satisfying

(1) $|\mathcal{U}^*|$ is a priori bounded in the size of φ^*; in particular, this bound is expressed by

$$\log |\mathcal{U}^*| \leq 3T(n-1)$$

where $T(n-1)$ is the number of variables of φ^*.

(2) $\mathcal{U}^* \subseteq \mathcal{U}$.

It is fairly easy to see that by putting

$$M^*A^n = \{M^*A^{n-1}_{I^{n-1}} : I^{n-1} \in \mathcal{E}^n \cap MA^n\},$$

we obtain an assignment on the variables of type n which satisfies all conjuncts of φ but the conjuncts of type (pow).

In order to satisfy the conjuncts of type (pow) as well we proceed as follows.

As in (9), let $H \subseteq \{1, \ldots, k\}$ and let

$$\begin{aligned} \mathcal{S}^*_H &= \bigcap_{h \in H} pow(M^*A^{n-1}_h) \setminus \bigcup_{h \notin H} pow(M^*A^{n-1}_h) \\ \mathcal{T}_H &= \bigcap_{h \in H} MB^n_h \setminus \bigcup_{h \notin H} MB^n_h \end{aligned}$$

Notice that $\mathcal{T}_H = \mathcal{S}_H$ for any $H \subseteq \{1, \ldots, k\}$.

The following lemma holds.

LEMMA 6.1 *For all $H \subseteq \{1, \ldots, k\}$, $\mathcal{S}^*_H = \emptyset$ if and only if $\mathcal{T}_H = \emptyset$.*

Proof. Suppose first that $\mathcal{S}^*_H = \emptyset$ and suppose by contradiction that $\mathcal{T}_H \neq \emptyset$. Therefore, there exists a nonempty Venn region σ such that

(1) $\sigma \subseteq MB^n_h$ for all $h \in H$;

(2) $\sigma \cap MB^n_h = \emptyset$ for all $h \notin H$.

Let $I^{n-1} \in \sigma$ be one of the individuals of $\mathcal{E}^n$ above chosen from σ. So, $I^{n-1} \subseteq MA^{n-1}_h$ for all $h \in H$ and $I^{n-1} \not\subseteq MA^{n-1}_h$ for all $h \notin H$. Therefore, $M^*A^{n-1}_{I^{n-1}} \subseteq M^*A^{n-1}_h$ for all $h \in H$ and $M^*A^{n-1}_{I^{n-1}} \not\subseteq M^*A^{n-1}_h$ for all $h \notin H$. So, $M^*A^{n-1}_{I^{n-1}} \in \mathcal{S}_H$.

Conversely, let $\mathcal{T}_H = \emptyset$. Therefore $\mathcal{S}_H = \emptyset$ and by construction $\bigcap_{h \in H} M^*A^{n-1}_h \subseteq M^*A^{n-1}_{h'}$ for some $h' \notin H$, therefore $\mathcal{S}^*_H = \emptyset$. ■

We now extend M^* in the following way.

Call a Venn region σ^n *singleton* if and only if $\sigma^n = \{MA^{n-1}\}$ for some A^{n-1} singleton variable occurring in φ.

Put now,

$$\sigma^{n,*} = \{M^* A^{n-1}_{I^{n-1}} : I^{n-1} \in \sigma^n\} \tag{11}$$

Let also

$$\mathcal{R}^*_H = \mathcal{S}^*_H \setminus \{M^* A^{n-1} : A^{n-1} \text{ occurring in } \varphi^*\}. \tag{12}$$

Let $H \subseteq \{1, \ldots, k\}$ such that $\mathcal{R}^*_H \neq \emptyset$. Our goal is to put the elements of $\mathcal{R}^*_H$ in the sets $\sigma^{n,*}$ in such a way that

(1) the sets $\sigma^{n,*}$ remain pairwise disjoint;

(2) if σ^n is singleton so is $\sigma^{n,*}$ and

(3) $Un(\sigma^{n,*})$ remains unchanged.

If we can accomplish this we can certainly claim that the assignment M^* extended over the variables of φ of type n in the following way

$$M^* A^n = \bigcup_{\sigma^n \subseteq MA^n} \sigma^{n,*} \tag{13}$$

is a model for φ.

Let us consider all the various cases.

(case 1) $I_H \notin \{MA^{n-1} : A^{n-1} \text{ occurring in } \varphi\}$.

In this case, let $\sigma^{n,*}$ be the set to which $M^* A^{n-1}_{I_H}$ belongs and put

$$\sigma^{n,*} := \sigma^{n,*} \cup \mathcal{R}^*_H.$$

(case 2) $I_H = MA^{n-1}$ for some A^{n-1} not singleton variable occurring in φ.

As above, let $\sigma^{n,*}$ be the set to which $M^* A^{n-1}$ belongs and put

$$\sigma^{n,*} := \sigma^{n,*} \cup \mathcal{R}^*_H.$$

(case 3) $I_H = MA^{n-1}$ for some A^{n-1} singleton variable occurring in φ.

Two cases are possible

(case 3.1) There exists $\sigma^{n,*} \subseteq \mathcal{S}^*_H$ such that $Un(\sigma^{n,*}) = M^*A^{n-1}$ and $M^*A^{n-1} \notin \sigma^{n,*}$.

In this case we put

$$\sigma^{n,*} := \sigma^{n,*} \cup \mathcal{R}^*_H.$$

(case 3.2) There does not exist $\sigma^{n,*} \subseteq \mathcal{S}^*_H$ such that $Un(\sigma^{n,*}) = M^*A^{n-1}$ and $M^*A^{n-1} \notin \sigma^{n,*}$.

Therefore, for any $\sigma^{n,*} \subseteq \mathcal{S}^*_H$ we have that either $\sigma^{n,*} = \{M^*A^{n-1}\}$ or $Un(\sigma^{n,*}) \subset M^*A^{n-1}$. This obviously implies that for any $\sigma^n \subseteq \mathcal{S}_H$ we have that either $\sigma^n = \{MA^{n-1}\}$ or $Un(\sigma^n) \subset MA^{n-1}$.

Therefore, for any $\sigma^n \subseteq \mathcal{S}_H$ such that $Un(\sigma^n) \subset MA^{n-1}$ there exists a $S^{n-2} \in MA^{n-1}$ such that $Un(\sigma^n) \subseteq I_H \setminus \{S^{n-2}\}$ and $I_H \setminus \{S^{n-2}\} \in \mathcal{E}^n$. Moreover, $I_H \setminus \{S^{n-2}\} \in \mathcal{S}_H$ and therefore there exists $\sigma^{n,*} \subseteq M^*A^{n-1}$ such that $M^*A^{n-1}_{I_H \setminus \{S^{n-2}\}} \in \sigma^{n,*}$.

For any $\sigma^{n,*} \subseteq M^*A^{n-1}$ such that $\sigma^{n,*} \neq \{M^*A^{n-1}\}$, if $pow(Un(\sigma^{n,*})) \cap \mathcal{R}^*_H \neq \emptyset$ we do the following

$$\begin{aligned} \sigma^{n,*} &:= \sigma^{n,*} \cup \mathcal{R}^*_H \cap pow(Un(\sigma^{n,*})) \\ \mathcal{R}^*_H &:= \mathcal{R}^*_H \setminus pow(Un(\sigma^{n,*})) \end{aligned}$$

Since any element of $\mathcal{R}^*_H$ is contained in $M^*A^{n-1}_{I_H \setminus \{S^{n-2}\}}$ for some $I_H \setminus \{S^{n-2}\} \in \mathcal{S}_H$, we can conclude that all elements of $\mathcal{R}^*_H$ are put in some set $\sigma^{n,*}$.

Let us now compute what is the bound on the number of elements of $\mathcal{U}^*$.

We know that if $n = 3$ then $\log |\mathcal{U}^*| \leq 2m + \log m \leq 3m$, where m is the number of variables occurring in the formula, whereas for $n > 3$ $\log |\mathcal{U}^*| \leq 3T(n-1)$ where $T(n-1)$ is the number of variables of φ^*.

Notice that $T(n-1) \leq 2^m$ where m is the number of variables occurring in φ.

Therefore, as it is easy to see,

$$|\mathcal{U}^*| \leq \mathcal{O}\left(2^{2^{2^{\cdot^{\cdot^{2^m}}}}}\right)$$

where m is the total number of variables in the formula and the length of the exponential stack is $n-2$.

We can then conclude with the following theorem.

THEOREM 6.1 *For any integer $n \geq 3$, the class of formulae $nLSSPU$ has a solvable satisfiability problem.* ■

7 Final remarks and open problems

The decidability result for the theory $3LSSPU$ described in the preceding sections and its generalization to $nLSSPU$ for $n \geq 4$, suggest many interesting research problems.

In particular, we conjecture that the theory $nLSSPU$ can be extended, without disrupting decidability, by adding predicates such as $u\text{-}closed(A)$, $fu\text{-}closed(A)$, $i\text{-}closed(A)$, $fi\text{-}closed(A)$, ranging over the variables of sort $i \geq 2$ and with the intended meaning of

- $u\text{-}closed(A) \equiv A$ *is closed with respect to union;*
- $fu\text{-}closed(A) \equiv A$ *is closed with respect to finite union;*
- $i\text{-}closed(A) \equiv A$ *is closed with respect to intersection;*
- $fi\text{-}closed(A) \equiv A$ *is closed with respect to finite intersection.*

Successively, one may try to introduce in the language functional operators such as

- $u - closure(A)$ or $i - closure(A)$ which for A ranging over the variables of sort $i \geq 2$ give respectively the sets obtained by closing A with respect to union and intersection;
- $sup(A, X)$ which for A ranging over the variables of sort $i \geq 2$ and X ranging over the variables of sort $i - 1$ gives the set of all elements of A that contain X,
- etc.

Finally, one can also try to introduce in the language the Kuratowski closure operator therefore incrementing the the collection of results described in [Cut91].

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