A Constructive $\Omega(t^{1.26})$
Lower Bound for the
Ramsey Number $R(3, t)$

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Abstract

We present a feasibly constructive proof that $R(3, t) > 5 \left( t^{1-1/\log^2 3} \right) \in \Omega(t^{1.26})$. This is, as far as we know, the first constructive superlinear lower bound for $R(3, t)$. Also, our result yields the first feasible method for constructing triangle-free $k$-chromatic graphs that are polynomial-size in $k$.

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1 Introduction

The Ramsey number $R(s, t)$ is the smallest integer for which every graph on $R(s, t)$ vertices contains either a clique of size $s$ or an independent set of size $t$. Ramsey (1930) shows that, for all $s$ and $t$, $R(s, t)$ is well defined. The determination of $R(s, t)$ has proven to be extremely difficult for all but a few values of $s$ and $t$.

Several upper and lower bounds on $R(s, t)$ for various values of $s$ and $t$ are known. Many of the lower bounds involve nonconstructive methods. These methods (introduced by Erdős (1947)) establish the existence of a graph of size $n(s, t)$ with clique size $s$ and independence number $t$ (which implies $R(s, t) > n(s, t)$), but they yield no feasible method for constructing such a graph.

The best currently known lower bound for $R(3, t)$ uses this approach and is

$$R(3, t) \geq \frac{1}{27} \left( \frac{t}{\log t} \right)^2$$

for sufficiently large $t$ (Spencer, 1977). The best known upper bound is

$$R(3, t) \leq \frac{t^2}{\log t}$$


Constructive lower bounds for $R(s, t)$ (that is, proofs that establish the existence of the desired graphs by providing a feasible construction for them) appear to be much more difficult to obtain. Many constructive lower bounds are known, but they are considerably weaker than their nonconstructive counterparts. As far as we know, the best previously known constructive lower bound for $R(3, t)$ is $R(3, t) \in \Omega(t)$.

We present a constructive proof that

$$R(3, t) > 5 \left( \frac{t - \frac{1}{2}}{2} \right)^{\log_3 4} \in \Omega(t^{1.26}).$$

Our result is based on a method that transforms any graph into another graph that has four times as many vertices, but whose independence number only increases by a factor of three. It is important that this transformation preserves the property of triangle-freeness.
In addition to the connection with Ramsey theory, our result contributes to the problem of feasibly constructing small triangle-free graphs with large chromatic numbers. The best previous results (Mycielski, 1955; Lovász, 1968; Nešetřil and Rödl, 1979) feasibly construct a triangle-free $k$-chromatic graph of size $O(2^k)$. Our result feasibly constructs a triangle-free $k$-chromatic graph of size $O(k^{\frac{\log^4 k}{\log \log k}}) \subset O(k^{4.82^k})$.

In Section 2, we review previously obtained bounds (both constructive and nonconstructive) for Ramsey numbers. In Section 3, we present our constructive lower bound.

2 Overview of Previous Results

It is easy to show that, for all $t$, $R(1,t) = 1$ and $R(2,t) = t$. Greenwood and Gleason (1955) show that $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$ and $R(4,4) = 18$, Kalbfleisch (1966) shows that $R(3,6) = 18$, Graver and Yackel (1968) show that $R(3,7) = 23$, and Grinstead and Roberts (1983) show that $R(3,9) = 36$. It has been suggested that the determination of $R(5,5)$ may be intractable (Gardner, 1977; Erdős, 1985), although it is known that $38 \leq R(5,5) \leq 67$.

Erdős (1947) introduces a nonconstructive method for proving lower bounds on $R(s,t)$ (this method has come to be known as “the probabilistic method”). The method involves assigning a probability distribution to the set of all graphs of size $n$, and proving that the probability is greater than zero that a graph chosen according to this distribution has neither a clique of size $s$ nor an independent set of size $t$. Although this method establishes the existence of a graph of size $n$ with clique size and independence number greater than $s$ and $t$ (respectively), it provides no feasible method for constructing such a graph.

The first result from the probabilistic method (Erdős, 1947) relates to the so-called “diagonal” Ramsey numbers $R(t,t)$. The result is that

$$R(t,t) \geq \frac{1}{e} \cdot t \cdot 2^\frac{\log t}{4}$$

(Spencer (1975) improves this lower bound by a multiplicative factor of 2). An upper bound of

$$R(t,t) < \frac{1}{6} \sqrt{t} \cdot 4^t$$
is well known (Skolem, 1933; Erdős and Szekeres, 1935).

Probabilistic methods also yield some interesting lower bounds for "off-diagonal" Ramsey numbers. Erdős (1961) proves that

\[ R(3, t) \in \Omega \left( \left( \frac{t}{\log t} \right)^2 \right) \]

and Spencer (1977) extends this to, for any fixed \( s \),

\[ R(s, t) \in \Omega \left( \left( \frac{t}{\log t} \right)^{\frac{s+1}{2}} \right) \]

In particular, for \( s = 3 \), Spencer's result is

\[ R(3, t) \geq \frac{1}{27} \left( \frac{t}{\log t} \right)^2 \]

for sufficiently large \( t \).

Define a feasibly constructive \( n(s, t) \) lower bound for \( R(s, t) \) as an algorithm that, on input \( s \) and \( t \) (from some pre-specified domain), explicitly constructs a graph of size \( n(s, t) \), in time polynomial in \( n(s, t) \), such that the graph contains no clique of size \( s \) and no independent set of size \( t \). Though nontrivial constructive lower bounds for \( R(s, t) \) are known, they appear much more difficult to obtain than their nonconstructive counterparts.

Abbott (1972) presents the first nontrivial constructive lower bound, showing that

\[ R(t, t) \in \Omega \left( t^{\log t} \right) \subset \Omega(t^{2.67}) . \]

This is improved by Nagy (1975) to

\[ R(t, t) \in \Omega(t^3) \]

and Frankl (1977) constructively proves that

\[ R(t, t) \in \Omega(t^k) \]

for all \( k \). Chung (1981) constructively proves that

\[ R(t, t) \geq \exp \left( \frac{\log t}{(\log \log t)^{1/3}} \right) \]
for some $c > 0$. Currently, the best constructive lower bound for the diagonal Ramsey numbers (due to Frankl and Wilson (1981)) is

$$R(t, t) \geq \exp \left( \frac{(1 - \epsilon)(\log t)^2}{4 \log \log t} \right)$$

for all $\epsilon > 0$ and sufficiently large $t$.

The only constructive superlinear lower bound for the off-diagonal Ramsey numbers known to the authors is a result of Alon (1986) that implies a constructive proof of

$$R(4, t) \in \Omega(t^{\frac{4}{3}}) \subset \Omega(t^{1.33}).$$

### 3 New Result

In this section, we define a method that transforms a graph $G$ to a graph consisting of four disjoint copies of $G$ connected by additional edges in a particular way. We then prove that this construction preserves the triangle-freeness property and increases the independence number, $\alpha$, of the graph by a factor of three, whereas, it increases the size of the graph by a factor of four. We conclude that, by repeatedly applying the transformation, a feasible construction of triangle-free graphs with independence number $t$ and size $\Omega(t^{\frac{\log 4}{\log 3}})$ is obtained. This constructively proves

$$R(3, t) \in \Omega \left( t^{\frac{\log 4}{\log 3}} \right) \subset \Omega(t^{1.26}).$$

**Definition 1** Let $G = (V(G), E(G))$. The fibration of $G$ is the graph $H = (V(H), E(H))$ with vertex set $V(H) = V(G) \times \{0, 1, 2, 3\}$ and edge set $E(H)$ joining $(u, i), (v, j) \in V(H)$ iff any one of the following conditions is satisfied: (i) $i = j$ and $(u, v) \in E(G)$; (ii) $i = 0$, $j = 3$ and $u = v$; or (iii) $i \in \{0, 1, 2\}$, $j = i + 1$ and $u \in N(v)$ where $N(v) = \{w : (v, w) \in E(G)\}$ is
the neighbourhood set of \( v \).

For \( i \in \{0, 1, 2, 3\} \) define the subsets \( V_i = \{(u, i) \in V(H)\} \). For any set \( U \subseteq V(H) \), let \( H[U] \) be the subgraph of \( H \) induced by \( U \). We define the projection \( \pi : V(H) \rightarrow V(G) \) by \( \pi(u, i) = u \) for all \( (u, i) \in V(H) \).

**Lemma 1** If \( H \) is the fibration of \( G \) and \( G \) is triangle-free then \( H \) is triangle-free.

**Proof** Let \( (u, i), (v, j) \) and \( (w, k) \) be any three vertices in \( H \). Clearly, if \( i, j \) and \( k \) are distinct, the three vertices cannot form a triangle in \( H \). Suppose \( i = j \) and, with out loss of generality, assume \( k = i + 1 \). If \((u, i)\) and \((v, i)\) are both joined to \((w, i + 1)\) in \( H \), by Definition 1, both \((u, i)\) and \((v, i)\) are also joined to \((w, i)\). Thus, if \((u, i), (v, i), (w, i + 1)\) form a triangle in \( H \) then \((u, i), (v, i), (w, i)\) form a triangle in the induced subgraph \( H[V_i] \). This is a contradiction since \( H[V_i] \) is isomorphic to \( G \). \( \square \)

**Lemma 2** If \( H \) is the fibration of \( G \) then \( |H| = 4 \cdot |G| \) and \( \alpha(H) = 3 \cdot \alpha(G) \).

**Proof:** Obviously, \( |H| = 4 \cdot |G| \). The interesting part is to show that \( \alpha(H) = 3 \cdot \alpha(G) \).

It may be helpful for the reader to make the following preliminary observation. If \( T \) is a maximal independent set of \( G \) then \( T \times \{0, 1, 2\} \) is a maximal independent set of \( H \). This implies that \( \alpha(H) \geq 3 \cdot \alpha(G) \) and that
one of the obvious approaches to constructing a large independent set in $H$
does not exceed the size of $3 \cdot \alpha(G)$.

Now, let $S$ be an arbitrary independent set in $H$. We show that $|S| \leq 3 \cdot \alpha(G)$. For $i \in \{0, 1, 2, 3\}$ let $S_i = \{(u, i) \in S\}$
We first show that for $i \in \{0, 1, 2\}$,

$$|S_i| + |S_{i+1}| \leq \alpha(G) + |\pi(S_i) \cap \pi(S_{i+1})|.$$

\text{(*)}

Fix $i \in \{0, 1, 2\}$. The equality

$$|\pi(S_i)| + |\pi(S_{i+1})| = |\pi(S_i) \cup \pi(S_{i+1})| + |\pi(S_i) \cap \pi(S_{i+1})|$$

follows trivially because set cardinality is a modular function. We prove that $\pi(S_i) \cup \pi(S_{i+1})$ is an independent set in the $G$. Thus, $|\pi(S_i) \cup \pi(S_{i+1})| \leq \alpha(G)$. From this, \text{(*)} follows because $|\pi(S_i)| = |S_i|$. It is obvious that both

$S_i$ and $S_{i+1}$ are independent sets in $H[V_i]$ and $H[V_{i+1}]$ and, therefore, $\pi(S_i)$ and $\pi(S_{i+1})$ are independent sets in $G$. Now assume some $u \in \pi(S_i)$ is

adjacent to some $v \in \pi(S_{i+1})$ in $G$. Then by Definition 1, there is an edge

in $E(H)$ joining $(u, i) \in S_i$ and $(v, i + 1) \in S_{i+1}$. This is a contradiction,

since $S$ is assumed to be independent.

Summing \text{(*)} with respect to all $i \in \{0, 1, 2\}$, we obtain

$$|S_0| + 2 \cdot |S_1| + 2 \cdot |S_2| + |S_3| \leq 3 \cdot \alpha(G) + |\pi(S_0) \cap \pi(S_1)| + |\pi(S_1) \cap \pi(S_2)| + |\pi(S_2) \cap \pi(S_3)|.$$  

Now, since $S$ is an independent set, by Definition 1 we require $\pi(S_3) \cap \pi(S_0) = \emptyset$, from which one can prove the purely set theoretic result that

$$|\pi(S_0) \cap \pi(S_1)| + |\pi(S_1) \cap \pi(S_2)| + |\pi(S_2) \cap \pi(S_3)| \leq |\pi(S_1)| + |\pi(S_2)|.$$  

Therefore,

$$|S| = |S_0| + |S_1| + |S_2| + |S_3| \leq 3 \cdot \alpha(G).$$

Thus $\alpha(H) \leq 3 \cdot \alpha(G)$. \square

\textbf{Theorem 3} There exists a feasible method for constructing a triangle-free graph with independence number less than $t$, whose size is $5(\frac{t-1}{2})^{\frac{\log 5}{2}}$. This constructively proves that $R(3, t) > 5(\frac{t-1}{2})^{\frac{\log 4}{2}} \in \Omega(t^{1.26})$.

\textbf{Proof} Construct a sequence of graphs $G_0, G_1, G_2, \ldots$ as follows. Let $G_0$ be a $5$-cycle, and let $G_{i+1}$ be the fibration of $G_i$. $G_0$ is triangle-free so, by
Lemma 1, for all $i$, $G_i$ is triangle-free. Clearly, $|G_0| = 5$ and $\alpha(G_0) = 2$. By Lemma 2, for all $i$, $|G_{i+1}| = 4 \cdot |G_i|$ and $\alpha(G_{i+1}) = 3 \cdot \alpha(G_i)$. Therefore, for all $i$, $|G_i| = 5 \cdot 4^i$ and $\alpha(G_i) = 2 \cdot 3^i$. If the sequence is constructed until $i = \log\left(\frac{t-1}{2}\right)/\log 3$ then $\alpha(G_i) = t - 1$ and $|G_i| = 5\left(\frac{t-1}{2}\right)^{\log 4/\log 3}$. □

References


