A Constructive $\Omega(t^{1.26})$ Lower Bound for the Ramsey Number $R(3,t)$

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Abstract

We present a feasibly constructive proof that $R(3,t) > 5(\frac{t-1}{2})^{\frac{\log 4}{\log 3}} \in \Omega(t^{1.26})$. This is, as far as we know, the first constructive superlinear lower bound for $R(3,t)$. Also, our result yields the first feasible method for constructing triangle-free k-chromatic graphs that are polynomial-size in k.

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1 Introduction

The Ramsey number $R(s,t)$ is the smallest integer for which every graph on $R(s,t)$ vertices contains either a clique of size s or an independent set of size t. Ramsey (1930) shows that, for all s and t, $R(s,t)$ is well defined. The determination of $R(s,t)$ has proven to be extremely difficult for all but a few values of s and t.

Several upper and lower bounds on $R(s,t)$ for various values of s and t are known. Many of the lower bounds involve nonconstructive methods. These methods (introduced by Erdös (1947)) establish the existence of a graph of size $n(s,t)$ with clique size s and independence number t (which implies $R(s,t) > n(s,t)$), but they yield no feasible method for constructing such a graph.

The best currently known lower bound for $R(3,t)$ uses this approach and is

$$R(3,t) \geq \frac{1}{27}\left(\frac{t}{\log t}\right)^2$$

for sufficiently large t (Spencer, 1977). The best known upper bound is

$$R(3,t) \leq \frac{t^2}{\log t}$$

for sufficiently large t (Ajtai, Komlós and Szemerédi, 1980;1981).

Constructive lower bounds for $R(s,t)$ (that is, proofs that establish the existence of the desired graphs by providing a feasible construction for them) appear to be much more difficult to obtain. Many constructive lower bounds are known, but they are considerably weaker than their non-constructive counterparts. As far as we know, the best previously known constructive lower bound for $R(3,t)$ is $R(3,t) \in \Omega(t)$.

We present a constructive proof that

$$R(3,t) > 5\left(\frac{t-1}{2}\right)^{\frac{\log 4}{\log 3}} \in \Omega(t^{1.26}).$$

Our result is based on a method that transforms any graph into another graph that has four times as many vertices, but whose independence number only increases by a factor of three. It is important that this transformation preserves the property of triangle-freeness.

In addition to the connection with Ramsey theory, our result contributes to the problem of feasibly constructing small triangle-free graphs with large chromatic numbers. The best previous results (Mycielski, 1955; Lovász, 1968; Nešetřil and Rödl, 1979) feasibly construct a triangle-free k-chromatic graph of size $O(2^k)$. Our result feasibly constructs a triangle-free k-chromatic graph of size $O(k^{\frac{\log 4}{\log 4 - \log 3}}) \subset O(k^{4.82})$.

In Section 2, we review previously obtained bounds (both constructive and nonconstructive) for Ramsey numbers. In Section 3, we present our constructive lower bound.

2 Overview of Previous Results

It is easy to show that, for all t, $R(1,t) = 1$ and $R(2,t) = t$. Greenwood and Gleason (1955) show that $R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$ and $R(4,4) = 18$, Kalbfleisch (1966) shows that $R(3,6) = 18$, Graver and Yackel (1968) show that $R(3,7) = 23$, and Grinstead and Roberts (1983) show that $R(3,9) = 36$. It has been suggested that the determination of $R(5,5)$ may be intractable (Gardner, 1977; Erdös, 1985), although it *is* known that $38 \leq R(5,5) \leq 67$.

Erdös (1947) introduces a nonconstructive method for proving lower bounds on $R(s,t)$ (this method has come to be known as "the probabilistic method"). The method involves assigning a probability distribution to the set of all graphs of size n, and proving that the probability is greater than zero that a graph chosen according to this distribution has neither a clique of size s nor an independent set of size t. Although this method establishes the existence of a graph of size n with clique size and independence number greater than s and t (respectively), it provides no feasible method for constructing such a graph.

The first result from the probabilistic method (Erdös, 1947) relates to the so-called "diagonal" Ramsey numbers $R(t,t)$. The result is that

$$R(t,t) \geq \frac{1}{e} \cdot t \cdot 2^{\frac{t-1}{2}}$$

(Spencer (1975) improves this lower bound by a multiplicative factor of 2). An upper bound of

$$R(t,t) < \frac{1}{6}\sqrt{t} \cdot 4^t$$

is well known (Skolem, 1933; Erdös and Szekeres, 1935).

Probabilistic methods also yield some interesting lower bounds for "off-diagonal" Ramsey numbers. Erdös (1961) proves that

$$R(3,t) \in \Omega\left(\left(\frac{t}{\log t}\right)^2\right)$$

and Spencer (1977) extends this to, for any fixed s,

$$R(s,t) \in \Omega\left(\left(\frac{t}{\log t}\right)^{\frac{s+1}{2}}\right).$$

In particular, for $s = 3$, Spencer's result is

$$R(3,t) \geq \frac{1}{27}\left(\frac{t}{\log t}\right)^2$$

for sufficiently large t.

Define a *feasibly constructive* $n(s,t)$ lower bound for $R(s,t)$ as an algorithm that, on input s and t (from some pre-specified domain), explicitly constructs a graph of size $n(s,t)$, in time polynomial in $n(s,t)$, such that the graph contains no clique of size s and no independent set of size t. Though nontrivial constructive lower bounds for $R(s,t)$ are known, they appear much more difficult to obtain than their nonconstructive counterparts.

Abbott (1972) presents the first nontrivial constructive lower bound, showing that

$$R(t,t) \in \Omega\left(t^{\frac{\log 41}{\log 4}}\right) \subset \Omega(t^{2.67}).$$

This is improved by Nagy (1975) to

$$R(t,t) \in \Omega(t^3)$$

and Frankl (1977) constructively proves that

$$R(t,t) \in \Omega(t^k)$$

for all k. Chung (1981) constructively proves that

$$R(t,t) \geq \exp\left(c\frac{(\log t)^{4/3}}{(\log\log t)^{1/3}}\right)$$

for some $c > 0$. Currently, the best constructive lower bound for the diagonal Ramsey numbers (due to Frankl and Wilson (1981)) is

$$R(t,t) \geq \exp\left(\frac{(1-\epsilon)(\log t)^2}{4\log\log t}\right)$$

for all $\epsilon > 0$ and sufficiently large t.

The only constructive superlinear lower bound for the off-diagonal Ramsey numbers known to the authors is a result of Alon (1986) that implies a constructive proof of

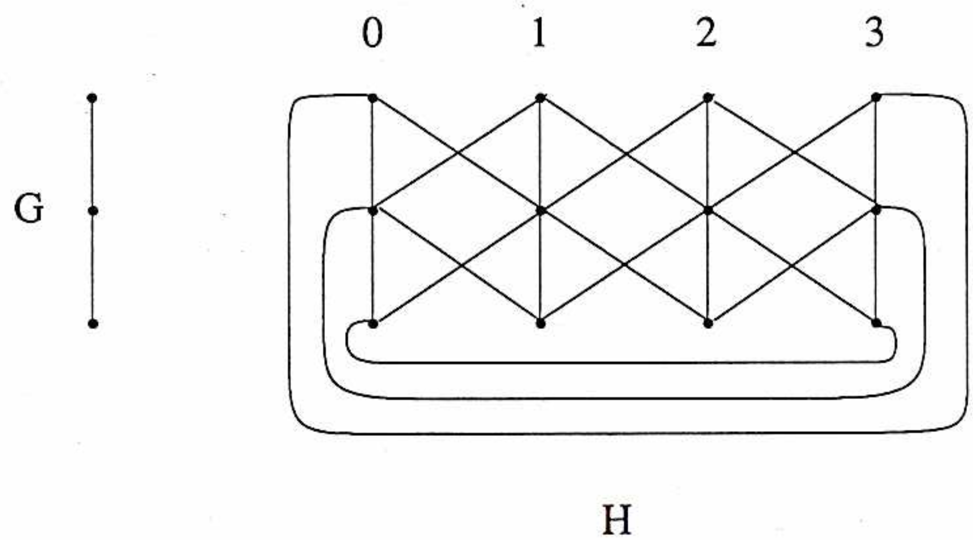
$$R(4,t) \in \Omega(t^{\frac{4}{3}}) \subset \Omega(t^{1.33}).$$

3 New Result

In this section, we define a method that transforms a graph G to a graph consisting of four disjoint copies of G connected by additional edges in a particular way. We then prove that this construction preserves the triangle-freeness property and increases the independence number, α, of the graph by a factor of three, whereas, it increases the size of the graph by a factor of four. We conclude that, by repeatedly applying the transfomation, a feasible construction of triangle-free graphs with independence number t and size $\Omega(t^{\frac{\log 4}{\log 3}})$ is obtained. This constructively proves

$$R(3,t) \in \Omega\left(t^{\frac{\log 4}{\log 3}}\right) \subset \Omega(t^{1.26}).$$

Definition 1 Let $G = (V(G), E(G))$. The *fibration* of G is the graph $H = (V(H), E(H))$ with vertex set $V(H) = V(G) \times \{0,1,2,3\}$ and edge set $E(H)$ joining $(u,i),(v,j) \in V(H)$ iff any one of the following conditions is satisfied: (i) $i = j$ and $(u,v) \in E(G)$; (ii) $i = 0$, $j = 3$ and $u = v$; or (iii) $i \in \{0,1,2\}$, $j = i+1$ and $u \in N(v)$ where $N(v) = \{w : (v,w) \in E(G)\}$ is the neighbourhood set of v.



For $i \in \{0,1,2,3\}$ define the subsets $V_i = \{(u,i) \in V(H)\}$. For any set $U \in V(H)$, let $H[U]$ be the subgraph of H induced by U. We define the projection $\pi : V(H) \rightarrow V(G)$ by $\pi(u,i) = u$ for all $(u,i) \in V(H)$.

Lemma 1 *If H is the fibration of G and G is triangle-free then H is triangle-free.*

Proof Let (u,i), (v,j) and (w,k) be any three vertices in H. Clearly, if i, j and k are distinct, the three vertices cannot form a triangle in H. Suppose $i = j$ and, with out loss of generality, assume $k = i+1$. If (u,i) and (v,i) are both joined to $(w,i+1)$ in H, by Definition 1, both (u,i) and (v,i) are also joined to (w,i). Thus, if $(u,i),(v,i)$, and $(w,i+1)$ form a triangle in H then $(u,i),(v,i)$, and (w,i) form a triangle in the induced subgraph $H[V_i]$. This is a contradiction since $H[V_i]$ is isomorphic to G. $\Box$

Lemma 2 *If H is the fibration of G then $|H| = 4 \cdot |G|$ and $\alpha(H) = 3 \cdot \alpha(G)$.*

Proof: Obviously, $|H| = 4 \cdot |G|$. The interesting part is to show that $\alpha(H) = 3 \cdot \alpha(G)$.

It may be helpful for the reader to make the following preliminary observation. If T is a maximal independent set of G then $T \times \{0,1,2\}$ is a maximal independent set of H. This implies that $\alpha(H) \geq 3 \cdot \alpha(G)$ and that one of the obvious approaches to constructing a large independent set in H does not exceed the size of $3 \cdot \alpha(G)$.

Now, let S be an arbitrary independent set in H. We show that $|S| \leq 3 \cdot \alpha(G)$. For $i \in \{0,1,2,3\}$ let $S_i = \{(u,i) \in S\}$

We first show that for $i \in \{0,1,2\}$,

$$|S_i| + |S_{i+1}| \leq \alpha(G) + |\pi(S_i) \cap \pi(S_{i+1})|. \qquad (*)$$

Fix $i \in \{0,1,2\}$. The equality

$$|\pi(S_i)| + |\pi(S_{i+1})| = |\pi(S_i) \cup \pi(S_{i+1})| + |\pi(S_i) \cap \pi(S_{i+1})|$$

follows trivially because set cardinality is a modular function. We prove that $\pi(S_i) \cup \pi(S_{i+1})$ is an independent set in the G. Thus, $|\pi(S_i) \cup \pi(S_{i+1})| \leq \alpha(G)$. From this, $(*)$ follows because $|\pi(S_i)| = |S_i|$. It is obvious that both S_i and S_{i+1} are independent sets in $H[V_i]$ and $H[V_{i+1}]$ and, therefore, $\pi(S_i)$ and $\pi(S_{i+1})$ are independent sets in G. Now assume some $u \in \pi(S_i)$ is adjacent to some $v \in \pi(S_{i+1})$ in G. Then by Definition 1, there is an edge in $E(H)$ joining $(u,i) \in S_i$ and $(v, i+1) \in S_{i+1}$. This is a contradiction, since S is assumed to be independent.

Summing $(*)$ with respect to all $i \in \{0,1,2\}$, we obtain

$$|S_0| + 2 \cdot |S_1| + 2 \cdot |S_2| + |S_3| \leq 3 \cdot \alpha(G) + |\pi(S_0) \cap \pi(S_1)| + |\pi(S_1) \cap \pi(S_2)| + |\pi(S_2) \cap \pi(S_3)|.$$

Now, since S is an independent set, by Definition 1 we require $\pi(S_3) \cap \pi(S_0) = \emptyset$, from which one can prove the purely set theoretic result that $|\pi(S_0) \cap \pi(S_1)| + |\pi(S_1) \cap \pi(S_2)| + |\pi(S_2) \cap \pi(S_3)| \leq |\pi(S_1)| + |\pi(S_2)|$. Therefore,

$$|S| = |S_0| + |S_1| + |S_2| + |S_3| \leq 3 \cdot \alpha(G).$$

Thus $\alpha(H) \leq 3 \cdot \alpha(G)$. □

Theorem 3 *There exists a feasible method for constructing a triangle-free graph with independence number less than t, whose size is $5(\frac{t-1}{2})^{\frac{\log 4}{\log 3}}$. This constructively proves that $R(3,t) > 5(\frac{t-1}{2})^{\frac{\log 4}{\log 3}} \in \Omega(t^{1.26})$.*

Proof Construct a sequence of graphs $G_0, G_1, G_2, \ldots$ as follows. Let G_0 be a 5-cycle, and let G_{i+1} be the fibration of G_i. G_0 is triangle-free so, by

Lemma 1, for all i, G_i is triangle-free. Clearly, $|G_0| = 5$ and $\alpha(G_0) = 2$. By Lemma 2, for all i, $|G_{i+1}| = 4 \cdot |G_i|$ and $\alpha(G_{i+1}) = 3 \cdot \alpha(G_i)$. Therefore, for all i, $|G_i| = 5 \cdot 4^i$ and $\alpha(G_i) = 2 \cdot 3^i$. If the sequence is constructed until $i = \log(\frac{t-1}{2})/\log 3$ then $\alpha(G_i) = t - 1$ and $|G_i| = 5(\frac{t-1}{2})^{\frac{\log 4}{\log 3}}$. $\square$

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