On the Decidability Problem for A Topological Syllogistic Involving the Notion of Topological Product

Domenico Cantone\(^1\)
Vincenzo Cutello\(^2\)

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Abstract

A two-level, multi-sorted language of sets with cartesian product is introduced. The solvability of the satisfiability problem for the corresponding class of unquantified formulae is shown to be useful in order to automatically verify the validity of certain topological statements involving the notion of product of spaces.

The underlying motivation for this study is to enrich the class of theoretical results that can be used for a set-theoretic proof verifier.

\(^1\)University of L’Aquila and University of Catania, Italy.

\(^2\)International Computer Science Institute, Berkeley, California and University of Catania, Italy.
1 Introduction

Many negative results concerning undecidability of quantified topological theories have been discovered.

In [11] it is proved that any algebra of topology is undecidable. Briefly, an algebra of topology is a first-order theory with set variables, boolean operations of union, intersection and set difference, the special constants $\emptyset$ and 1 and where the non-logical axioms are specified by a certain topological notion $\phi$ which must be consistent with Kuratowski's closure axioms for a metrizable topological space.

This in particular implies that any fully quantified two level syllogistic that includes either Kuratowski's closure operator, or the predicates closed or open is undecidable.

Many other decidability and undecidability results can be found in [13, 1, 12]. These results concern however more restricted topological decision problems that relates to some group theoretical word problems. So decidability or undecidability is a consequence of solvability or unsolvability of certain group properties (cf. [8]).

We study a different aspect of the problem. Specifically, we concentrate on elementary abstract topology, i.e. on the problem of deciding if there exists a topological space and certain subsets of its that satisfy a given formula. Our results follow on from those of [5, 6].

The specific long term goal that we aim to assist is the building of a powerful computerized proof verifier. The power of such verifiers will be directly related to the richness of their inferential core, i.e. to the collection of modules in them able to make bottom-level deduction steps in specific areas of Mathematics. Guided by this observation, work in the last decade has focused on and produced a significant collection of decidability results for fragments of set theory (cfr. [4]). This emphasis on set theory is justified by the familiar observation that any mathematical theory can be embedded in set theory.

2 Two Level Syllogistic and the cartesian product

By Two Level Syllogistic (2LS) we mean the two sorted unquantified set-theoretic language obtained as the propositional closure of atoms of type

- $X = Y \cup Z$;
- $X = Y \setminus Z$;
- $X = Y \cap Z$;
- $x \in X$;
- $x = y$. 

Here, capital letters denote set variables and small letters denote element variables. The satisfiability problem for this language has been studied in [10].

For our purposes, it is important to note that the extension of $2LS$ with singleton and cardinality comparison clauses of type

$$X = \{x\}, \quad |X| \leq |Y|, \quad |X| < |Y|$$

(which we denote by $2LSSC$) also has a solvable satisfiability problem. Indeed, the language $2LSSC$ can be embedded in the language $MLS$ extended by singleton and cardinality comparison, whose decidability is known (cf. [2] and [9]).

### 2.1 The cartesian product

It is still an open problem whether $MLS$ extended by the cartesian product operator has a solvable satisfiability problem or not. However we can extend $2LS$ by means of such an operator and obtain decidability. We shall now describe such a language, denoted $\text{Cart}^{2,\infty}$, in detail.

$\text{Cart}^{2,\infty}$ is a two level multi-sorted language which contains for each natural number $p$,

- an infinite supply of individual variables, $x_n^{(p)}$ standing for element variables of sort $p$;
- an infinite supply of individual variables, $X_n^{(p)}$ standing for set variables of sort $p$;

Moreover, for each $p$ the constant $\emptyset^{(p)}$ is in our language.

$\text{Cart}^{2,\infty}$ properly extends $2LS$ by endowing the variables with a sort and by allowing occurrences of atoms of type

- $X = \{x\}$;
- $X = Y \uplus Z$.

The following syntactic restrictions are placed on the atoms of $\text{Cart}^{2,\infty}$:

- the binary operators $\cup$, $\setminus$, $\cap$ can be applied to set variables of the same sort and their result is a set variable of the same sort;
- the binary predicate $=, \in$ can be applied to variables of the same sort;
- the unary operator $\{\cdot\}$ is applied to an element variable and its result is a set variable of the same sort as the element variable;
- the $\uplus$ operator can be applied to set variables of different sort, say $p_1, p_2$, and its result is a set variable of sort $p_1 + p_2$. 

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**Remark 2.1** The intuition behind the above discussion is the following. All the set variables of sort 1 represent sets whose elements have no internal structure. The cartesian operator, when applied to two set variables of sort 1, produces a set whose elements are ordered list of two elements of sort 1. Thus, in general, a set variable of sort $p$ will represent a homogeneous set whose elements are all ordered list of $p$ elements of sort 1.

**Remark 2.2** The projection functions $\pi_q(y^{(p)})$ which for $q \leq p$ return the $q$-th element of $y^{(p)}$, are expressible in $Cart^{2,\infty}$ for all $q = 1, \ldots, p$. Indeed, $x^{(1)} = \pi_q(y^{(p)})$ is equisatisfiable with

(a) if $q = 1$:

$$X^{(1)} = \{x^{(1)}\} \land y^{(p)} \in Y^{(p)} \land Y^{(p)} = X^{(1)} \otimes Z^{(p-1)}.$$ 

(b) if $q = p$

$$X^{(1)} = \{x^{(1)}\} \land y^{(p)} \in Y^{(p)} \land Y^{(p)} = Z^{(p-1)} \otimes X^{(1)}.$$ 

(c) if $1 < q < p$

$$X^{(1)} = \{x^{(1)}\} \land y^{(p)} \in Y^{(p)} \land Y^{(p)} = Z^{(p-2)} \otimes Z^{(p-1)} \land X^{(1)} = Z^{(p-1)} \otimes X^{(1)}.$$ 

Using a standard normalization process the satisfiability problem for the theory $Cart^{2,\infty}$ can be reduced to the satisfiability problem for conjunctions, that we will call normalized, of literals of type:

- $X^{(i)} = Y^{(i)} \cup Z^{(i)}$, $X^{(i)} = Y^{(i)} \setminus Z^{(i)}$
- $x^{(i)} = y^{(i)}$
- $X^{(i)} = \{x^{(i)}\}$
- $X^{(i)} = Y^{(i)} \otimes Z^{(h)}$, where $i = j + h$.

The solvability of the satisfiability problem for $Cart^{2,\infty}$ was shown in [3], along with the $\mathcal{NP}$-completeness of this problem. This result was derived from the decidability of a purely universally quantified theory of relations that properly extends it (see also [7] for a direct proof of it).

In the next sections we will show how these results can be used to prove that the satisfiability problem for a certain topological class of unquantified formulae is solvable.
3 The topological language \( \mathcal{L}_{p}^{2,\infty} \)

The topological language that we will describe, to be denoted by \( \mathcal{L}_{p}^{2,\infty} \), allows the notion of product spaces and, as a by-product, the cartesian product of sets. \( \mathcal{L}_{p}^{2,\infty} \) is the extension of \( \mathcal{C}art^{2,\infty} \) obtained by introducing the Kuratowski closure operator.

Specifically, \( \mathcal{L}_{p}^{2,\infty} \) is a two level multi-sorted language which, for each natural number \( p \) contains

- an infinite supply of individual variables \( x_{0}^{(p)}, \ldots, x_{h}^{(p)}, \ldots, y_{0}^{(p)}, \ldots \) standing for elements of a topological space with sort \( p \). The collection of all these variables will be called \( \text{Indiv.var}(p) \).

- an infinite supply of set variables \( X_{0}^{(p)}, \ldots, X_{h}^{(p)}, \ldots, Y_{0}^{(p)}, \ldots, Y_{h}^{(p)}, \ldots \) standing for subsets of a topological space of the form \( 1 \otimes 1 \otimes \cdots 1 \) (\( p \) factors) where \( 1 \) represents some given basic topological space. The collection of all these variables will be called \( \text{Set.var}(p) \).

- the constants \( 0^{(p)} \) and \( 1^{(p)} \) denoting respectively the empty set and the topological space \( 1 \otimes 1 \otimes \cdots 1 \) (\( p \) factors).

Remark 3.1 By referring to the sort of a topological space we mean to imply the existence of a topological space denoted by \( 1 \) (or \( 1^{(1)} \)) such that \( 1^{(p)} = \otimes_{q=1}^{p}1 \), as above. Thus, each element of \( 1^{(p)} \) will be a \( p \)-uple of elements of \( 1^{(1)} \) whereas the elements of \( 1^{(1)} \) will be seen as atoms. Then any set variable \( X^{(p)} \) of sort \( p \) represents a set of \( p \)-uples, and any element variable \( x^{(p)} \) represents a \( p \)-uple. \( \square \)

Operators which allow one to build compound terms are also present in \( \mathcal{L}^{2,\infty} \). These operators are \( \cup, \cap, \setminus, ' \) (complementation), \( \sim \) (Kuratowski’s topological closure), \( \otimes, \{ \} \) (singleton) and projection functions \( \pi_{q} \).

More precisely, the terms of \( \mathcal{L}_{p}^{2,\infty} \) are defined as follows.

DEFINITION 3.1 For each \( p \)

- each variable in \( \text{Indiv.var}(p) \) is an individual term of sort \( p \);

- each variable in \( \text{Set.var}(p) \) and the special constants \( 1^{(p)}, 0^{(p)} \) are set terms of sort \( p \);

- if \( T_{1}, T_{2} \) are set terms of sort \( p \) and \( t_{1} \) is an individual term of sort \( p \) then \( T_{1} \cup T_{2}, T_{1} \cap T_{2}, T_{1} \setminus T_{2}, (T_{1})', T_{1}, \{ t_{1} \} \) are set terms of sort \( p \);

- if \( T_{1}, T_{2} \) are set terms of sorts respectively \( p \) and \( q \) then \( T_{1} \otimes T_{2} \) is a set term of sort \( p + q \) (cfr. Remark 3.1).
• if \( t_1 \) is an individual term of sort \( p \geq 2 \) then for each \( q = 1, \ldots, p \) \( \pi_q(t_1) \) is an individual term of sort \( 1 \).

The atomic formulae of \( \mathcal{L}_p^{2,\infty} \) are

\[
T_1 = T_2, T_1 \subseteq T_2, t_1 \in T_1,
\]

where \( T_1, T_2 \) are set terms of sort \( p \), \( t_1 \) is an individual term of sort \( p \).

The theory \( \mathcal{L}_p^{2,\infty} \) can then be defined as the propositional closure of atomic formulae of the above types by means of the logical connectives \( \lnot, \land, \lor, \rightarrow, \leftrightarrow \).

**Definition 3.2** A topological assignment \( M \) is any interpretation of the constants and variables of \( \mathcal{L}_p^{2,\infty} \) such that

1. \( M1^{(1)} \) is a topological space with topology \( \tau^{(1)} \) and \( M0^{(1)} \) is the empty set.
2. For any \( p \geq 2 \), \( M1^{(p)} = \bigotimes_{q=1,\ldots,p} M1^{(1)} \) endowed with a topology \( \tau^{(p)} \) which is the product topology \( \bigotimes_{q=1,\ldots,p} \tau^{(1)} \) and \( M0^{(p)} \) is the empty set.
3. \( Mx \in M1^{(p)} \) for each individual variable \( x \) in Indiv.var(\( p \));
4. \( MX \subseteq M1^{(p)} \) for each set variable \( X \) in Set.var(\( p \));
5. \( ' \) is interpreted as the set complementation in \( M1^{(p)} \), for any \( p \geq 1 \).
6. \( - \) is interpreted as the Kuratowski closure operator in the topological space \( (M1^{(p)}, \tau^{(p)}) \), for any \( p \geq 1 \).
7. \( \cup, \cap, \setminus \) are interpreted in the standard way;
8. the operator \( \otimes \) is interpreted as the cartesian product. So, if \( MX^{(n)} = MY_1^{(2)} \otimes MY_2^{(3)} \) then \( MX^{(n)} \) will be a set of \( q_1 \)-uples where the first \( q_2 \) elements of any \( q_1 \)-uple are the elements of a \( q_2 \)-uple in \( MY_1^{(2)} \) and the remaining \( q_3 \) elements are the elements of a \( q_3 \)-uple in \( MY_2^{(3)} \);
9. the projection operators \( \pi_q \) applied to elements of any topological space \( M1^{(p)} \) with \( p \geq q \geq 2 \) are interpreted in the usual way. So \( \pi_q(My^{(p)}) \) gives the \( q \)-th element of the \( p \)-uple \( My^{(p)} \);
10. the predicates \( =, \subseteq, \in \), are interpreted in the obvious way.

A formula \( \varphi \) of \( \mathcal{L}_p^{2,\infty} \) is said (topologically) satisfiable if there exists a topological assignment \( M \) which satisfies it.

Moreover, if \( M1^{(1)} \) is a finite set then \( \varphi \) is said (topologically) finitely satisfiable. \( \square \)
3.1 Normalized conjunctions

By means of a simple normalization process the satisfiability problem for the theory $\mathcal{L}_P^{2,\infty}$ can be reduced to the satisfiability problem for normalized conjunction of literals of type:

(i) $X^{(p)} = Y^{(p)} \cup Z^{(p)}$, $X^{(p)} = Y^{(p)} \setminus Z^{(p)}$;

(ii) $X^{(1)} = \{x^{(1)}\}$;

(iii) $x^{(p)} = y^{(p)}$;

(iv) $X^{(p)} = Y^{(r)} \otimes Z^{(r)}$ where $r + q = p$.

(v) $X^{(p)} = \overline{Y^{(p)}}$.

Indeed,

- $x \neq y$ is equisatisfiable with $Z_x = \{x\} \land Z_y = \{y\} \land Z_x \land Z_y$;
- $X \subseteq Y$ is logically equivalent to $X \setminus Y = 0$;
- $X = Y \cap Z$ is logically equivalent to $X = Y \setminus (Y \setminus Z)$;
- $Y = (X)'$ is logically equivalent to $Y = 1 \setminus X$;
- $x \in X$ is equisatisfiable with $Z_x = \{x\} \land Z_x \subseteq X$;
- $x \notin X$ is equisatisfiable with $Z_x = \{x\} \land Z_x \cap X = 0$;
- $X \neq Y$ is equisatisfiable with $(x_{XY} \in X \land x_{XY} \notin Y) \lor (x_{XY} \notin X \land x_{XY} \in Y)$;
- the projection operators can clearly be eliminated since they are expressible in the language $Cart^{2,\infty}$ (see Remark 2.2),

where the variables $Z_x, Z_y, x_{XY}, A_X, B_X, x_1, x_2$ are newly introduced variables of the appropriate type and sort.

4 Safe variables

Solvability of the general satisfiability problem for formulae of the class $\mathcal{L}_P^{2,\infty}$ is still an open problem. We can however solve the satisfiability problem in a useful subcase.

We recall that one of the goals of our research work is to obtain satisfiability tests that could provide a proof verifier with some means to quickly accomplish certain simple deduction steps that may be done quite expensively by a resolution based theorem prover.

Consider for instance the following simple statement:
if $X = Y \otimes Z$ and $y$ and $z$ are accumulation points of $Y$ and $Z$ respectively then the ordered pair $x = (y, z)$ is an accumulation point for $X$.

A proof of its validity by means of a resolution based theorem prover may be too expensive, both in terms of computer time and space (cf. [7]). The decidability result that we shall describe in this paper will be able to obtain this proof very fastly, considering the low number of variables involved.

Let us introduce the following notion.

Let $\Phi$ be a normalized conjunction of $L^{2,\infty}_p$ and let $p$ be the maximum sort in $\Phi$. Let for $q = 1, \ldots, p$, $V_q^p$ be the collection of set variables of sort $q$ occurring in $\Phi$.

Note that given a formula $\Phi$ we can suppose that the singleton operator $\{\cdot\}$ applies only to variables of sort 1. Indeed, $Y^{(q)} = \{y^{(q)}\}$ is equisatisfiable with $X_1 = \{x_1\} \land \cdots \land X_q = \{x_q\} \land Y^{(q)} = \otimes_{r=1,\ldots,q} X_r$.

**DEFINITION 4.1** A set variable $X^{(q)} \in V_q^q$ is safe if either

(s1) $q = 1$ i.e. all set variables of sort 1 are safe; or

(s2) a literal $X^{(q)} = Y^{(u)} \otimes Z^{(v)}$ is in $\Phi$ for some $Y^{(u)} \in V_u^u$, $Z^{(v)} \in V_v^v$, with $u + v = q$ and $u, v \neq q$ or

(s3) a literal $X^{(q)} = Y^{(q)} \cup Z^{(q)}$ or $X^{(q)} = Y^{(q)} \setminus Z^{(q)}$ or $X^{(q)} = \overline{Y^{(q)}}$ is in $\Phi$ for some $Y^{(q)}$, $Z^{(q)} \in V_q^q$, with $Y^{(q)}$ and $Z^{(q)}$ safe.

We say that $\Phi$ is safe if all the set variables of sort $q > 1$ occurring in $\Phi$ are safe.

Note that safeness of a variable relates its possible model to that of certain sets of smaller sort.

From Definition 4.1 it is immediate that we can define a function $S(Y)$ that defines a degree of safeness for each safe variable $Y$, as explained below.

1. if $Y = X_1 \otimes X_2$ then $S(Y) = 0$. Recursively, then,

2. $S(Y) \leq \max(S(Y_1), S(Y_2)) + 1$ if $Y_1 = Y_2 \ast Y_3$ is in $\Phi$ with $\ast = \cup$ or $\ast = \setminus$ and $S(Y_1) \leq S(Y_2) + 1$ if $Y_1 = \overline{Y_2}$ is in $\Phi$.

We will show in the next subsection that the satisfiability problem for safe, normalized conjunctions of $L^{2,\infty}_p$ can be reduced to the satisfiability problem for a more restricted class of conjunctions, namely a class of conjunctions where Kuratowski's closure operator and set difference operator apply only to variables of sort 1.
4.1 Reduction to a more restricted case

Let $\Phi$ be a safe, normalized conjunction and let $M$ be a model for $\Phi$. Then, for each $q = 1, \ldots, p$, $(M^{(q)}, \tau^{(q)})$ is a topological space. Suppose in particular that $\tau^{(q)}$ is the collection of closed sets of the topology over $M^{(q)}$.

Let us start first considering the simple case in which $q = 2$. For simplicity we will denote with $X_1, X_2, \ldots$ and $x_1, x_2, \ldots$ and with $Y_1, Y_2, \ldots$ and $y_1, y_2, \ldots$ respectively the set and element variables of type 1 and 2.

**Lemma 4.1** Let $\Phi$ be a safe formula of $\mathcal{L}_p^{2,\infty}$ such that the maximum sort of variables occurring in $\Phi$ is 2. Let $V^1_2$ and $V^2_2$ be respectively the collection of set variables of sort 1 and 2 occurring in $\Phi$. Then for each $Y$ set variable of sort 2 in $\Phi$ one can introduce suitably many auxiliary variables $F^{(2)}_{Y_i}$ and $A^{(1)}_{Y_i}, B^{(1)}_{Y_i}$, and suitably many conjuncts $C$ involving only set variables of sort 1, such that, called $\Phi^-$ the formula obtained from $\Phi$ when all conjuncts involving $Y$ of type $Y = \overline{Y}$ or $Y = Y_1 \setminus Y_2$ have been eliminated, we have that $\Phi^- \land (\land_{j<\ell} C_j \land \land_{i<\ell} (F^{(2)}_{Y_i} = A_{Y_i} \otimes B_{Y_i}))$ is equisatisfiable with $\Phi$.

**Proof.** We proceed by induction on $k = \max (\{S(Y) : Y \in V^2_2\})$.

**Basis case:** if $k = 0$ then for each $Y \in V^2_2$ a conjunct of type $Y = X_1 \otimes X_2$ is in $\Phi$.

- Suppose there is a clause $Y_3 = Y_1 \setminus Y_2$ which belongs to $\Phi$; we must show how to eliminate it. Since $S(Y_3) = S(Y_2) = S(Y_1) = 0$ there exist set variables $X_{ij}$ of sort 1, $i = 1, 2, 3$ and $j = 1, 2$ such that $Y_i = X_{i1} \otimes X_{i2}$, is in $\Phi$ for $i = 1, 2, 3$. So, $Y_3 = (X_{11} \setminus X_{21}) \otimes X_{12} \cup X_{11} \otimes (X_{12} \setminus X_{22})$. This observation allows the conjunct $Y_3 = Y_1 \setminus Y_2$ to be eliminated by introducing two set variables of sort 1, $X, X'$ and two set variables of sort 2, $Y, Y'$ and by putting:

$$X = X_{11} \setminus X_{21} \land X' = X_{12} \setminus X_{22} \land Y = X \otimes X_{12} \land Y' = X_{11} \otimes X' \land Y_3 = Y \cup Y'.$$

- Suppose that $Y_2 = \overline{Y}$ is in $\Phi$. Again, since $Y_2 = X_{21} \otimes X_{22}$ and $Y_1 = X_{11} \otimes X_{12}$, the conjunct can be eliminated by putting: $X_{21} = \overline{X}_{11}$ and $X_{22} = \overline{X}_{12}$.

**Inductive step:** suppose the assert is true for all $Y$ such that $S(Y) < k$ and let us prove it for $k$.

- Suppose that $Y_2 = \overline{Y}$ is in $\Phi$ with $S(Y_1) < S(Y_2) = k$. By inductive hypothesis $Y_1 = \bigcup_{i \in I} Z_i$ and for all $i \in I$ $Z_i = A_i \otimes B_i$. For all $i \in I$, we introduce new set variables $W_i, C_i, D_i$ and put

$$Y_2 = \bigcup_{i \in I} W_i \land (\land_{i \in I} W_i = C_i \otimes D_i \land (\land_{i \in I} (C_i = \overline{A_i} \land D_i = \overline{B_i})).$$

Finally we remove $Y_2 = \overline{Y}$ from $\Phi$. 

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• Suppose that \( Y_3 = Y_1 \setminus Y_2 \) is in \( \Phi \) with \( S(Y_1), S(Y_2) < S(Y_3) = k \). By inductive hypothesis

\[
Y_1 \setminus Y_2 = \bigcup_{i \in I} Z_i \setminus \bigcup_{j \in J} W_j = \bigcup_{i \in I} A_i \otimes B_i \setminus \bigcup_{j \in J} C_j \otimes D_j
\]

\[
= \bigcup_{i \in I} (A_i \otimes B_i \setminus \bigcup_{j \in J} C_j \otimes D_j)
\]

\[
= \bigcup_{i \in I, j \in J} (A_i \otimes B_i \setminus C_j \otimes D_j)
\]

\[
= \bigcup_{i \in I, j \in J} ((A_i \setminus C_j) \otimes B_i \cup A_i \otimes (B_i \setminus D_j))
\]

\[
= \bigcup_{i \in I} \bigcup_{j \in K \setminus j \in K} ((A_i \setminus C_j) \cap B_i \cap \bigcap_{j \in J \setminus K} (A_i \otimes (B_i \setminus D_j)))
\]

where \( \bigcap_{j \in J}(A_i \setminus C_j) \) stands for \( A_i \) and where \( \bigcap_{j \in J}(B_i \setminus D_j) \) stands for \( B_i \). At any rate, \( Y_3 = \bigcup_{i \in I, j \in K} W_{iK} \), with \( W_{iK} = E_{iK} \otimes F_j \) where \( E_{iK} = \bigcap_{j \in K}(A_i \setminus C_j) \) and \( F_{jK} = \bigcap_{j \in K}(B_i \setminus D_j) \), which can be easily expressed in terms of \( (\cup, \setminus) \)-formulae with all variables of sort 1.

• Thus we have proved that the satisfiability problem for safe formulae is equivalent to the satisfiability problem for safe conjunctions of the following kind of literals:

\[
Y_3 = Y_1 \cup Y_2, Y = X_1 \otimes X_2, X_1 = \overline{X_2}
\]

\[
X_3 = X_1 \cup X_2, X_3 = X_1 \setminus X_2, X = \{x\}.
\]

It is easy to see that this argument can be generalized to all \( q \geq 2 \). In this case we have safe conjunctions of the following kind of literals:

\[
Y_3^{(r)} = Y_1^{(r)} \cup Y_2^{(r)}, Y^{(r)} = X_1^{(r)} \otimes X_2^{(r)}, X_1^{(1)} = \overline{X_2^{(1)}}
\]

\[
X_3^{(1)} = X_1^{(1)} \cup X_2^{(1)}, X_3^{(1)} = X_1^{(1)} \setminus X_2^{(1)}, X^{(1)} = \{x^{(1)}\},
\]

The satisfiability problem for the restricted class of conjunctions above described follows as a consequence of the satisfiability problem of \( 2LS \) and cartesian product.

More specifically, given one such a conjunction \( \varphi \) we proceed as follows.

Let \( V^1 \) be the collection of set variables of sort 1. For each \( S \in \text{pow} (\text{pow}(V^1)) \) let \( C_S \) be a newly introduced set variables of sort 1. Then we add to \( \varphi \) the conjuncts

(F1) \( C_{\{x\}} = \overline{X} \);
(F2) \( \bigcap_{Q \in S} \bigcup_{X \in Q} C_{\{X\}} = C_S \) for any \( S \in \text{pow} \{\text{pow}(V^1)\} \);  
(F3) \( X \subseteq C_{\{X\}} \) for any \( X \in V^1 \);  
(F4) \( X \subseteq C_S \rightarrow C_{\{X\}} \subseteq C_S \) for any \( X \in V^1 \) and \( S \in \text{pow}(\text{pow}(V^1)) \).

and we drop from \( \phi \) all the conjuncts of type \( X = \overline{X} \).

The obtained formula \( \phi^* \) is a formula of \( \text{Cart}^{2,\infty} \).

We then need to prove the following Lemma.

**Lemma 4.2** \( \phi \) is topologically satisfiable if and only if \( \phi^* \) is satisfiable.

**Proof.** It is trivial to see that if \( \phi \) is satisfiable then \( \phi^* \) is satisfiable.

Conversely, suppose that \( \phi^* \) is satisfiable and let \( M^* \) be a model for it. We have to endow \( M^*1^{(1)} \) with a topology and so we define \( C = \{C_S : S \in \text{pow}(\text{pow}(V^1))\} \) as the class of closed sets of the topology. The conjuncts of type (F2) assure that \( C \) defines indeed a topology.

Then, for any \( i \geq 2 \) we define the topology on \( 1^{(i)} \) as the product topology of the topologies on \( 1^{(1)} \) and \( 1^{(i-1)} \).

Since all the set theoretic conjuncts are satisfied by definition, we need only to show that the conjuncts of type \( X = \overline{X} \) are satisfied.

From (F3) we have that \( M^*X_1 \subseteq M^*X = C_{\{X\}} \) and from (F4) we have that any other closed set in \( C \) that contains \( M^*X_1 \) must contain \( M^*X \), which proved that \( M^*X \) is the closure of \( M^*X_1 \).

We can then claim

**Theorem 4.1** The class of safe formulae has a solvable satisfiability problem.

5 Weak and finite satisfiability

The obvious extension of the class of safe formulae is given by the following definition.

**Definition 5.1** Let \( \varphi \) be a formula of \( \text{L}^{2,\infty}_p \). We say that \( \varphi \) is weakly satisfiable if there exists a model \( M \) such that for all \( Y \) of sort \( q \geq 2 \), \( MY \) is finite union of sets of type \( \bigotimes_{j=1,...,q} A_j \).

We will prove now that weak satisfiability and finite satisfiability are equivalent, i.e. that a normalized conjunction is weakly satisfiable if and only if it is finitely satisfiable. Clearly, finite satisfiability implies weak satisfiability, so we only have to prove the converse. To this end, we argue as follows.
Let \( \varphi \) be a normalized conjunction of \( \mathcal{L}_p^{2,\infty} \) and suppose \( p \) is the maximum sort of variables occurring in \( \varphi \). Suppose that \( \varphi \) is weakly satisfiable and let \( M \) be a weak model for \( \varphi \). Denote with \( V_\varphi^{(q)} \) the collection of set variables of sort \( q \) occurring in \( \varphi \) for any \( q = 1, \ldots, p \).

So, for any \( q \geq 1 \) there exists a collection \( \tau^{(q)} \) of subsets of \( M^{(q)} \) which is a topology over \( M^{(q)} \).

In particular, notice that \( M^{(q)} = \bigotimes_{r=1}^{q} M^{(r)} \) and \( \tau^{(q)} \) is the correspondent product topology, for any \( q = 2, \ldots, p \). So, for any \( q = 2, \ldots, p \), each closed set in \( \tau^{(q)} \) is intersection of finite unions of sets of type

\[
\bigotimes_{r=1}^{q} C_r
\]

where \( C_r \in \tau^{(r)} \) for any \( r = 1, \ldots, q \).

Notice now that by hypothesis, for any \( X^{(q)} \) set variable of sort \( q \) occurring in \( \varphi \), \( M X^{(q)} = \bigcup_{i \in I} \bigotimes_{j=1}^{q} A_{ij} \) where \( I \) is a finite set of indices and \( A_{ij} \subseteq M^{(1)} \) for any \( i \in I \) and \( j = 1, \ldots, q \).

Thus, starting from \( \varphi \) we can build a weak-safe normalized conjunction \( \varphi^* \) whose size is dependent on \( M \) in the following way.

For any \( X^{(q)} \in V_\varphi^{(q)} \), for any \( i \in I \) and for any \( j = 1, \ldots, q \) let \( Z_{ij} \) be a newly introduced set variable of sort 1.

Moreover, for any \( i \in I \), let \( W_i, W_{i1}, \ldots, W_{i,q-2} \) be newly introduced set variables of sort respectively, \( q, 2, \ldots, q - 1 \).

We put,

\[
W_{i1} = Z_{i1} \otimes Z_{i2} \text{ and for } r = 2, \ldots, q - 2
\]

\[
W_{ir} = W_{i,r-1} \otimes Z_{i,r+1} \text{ and}
\]

\[
W_i = W_{i,q-2} \otimes Z_{i,q}
\]

Then, we introduce \( k = |I| - 2 \) set variables of sort \( q \), say \( Q_1, \ldots, Q_k \) and we put

\[
Q_1 = W_1 \cup W_2 \text{ and for } j = 2, \ldots, |I| - 2
\]

\[
Q_j = Q_{j-1} \cup W_{j+1} \text{ and finally}
\]

\[
X^{(q)} = Q_k \cup W_{|I|}
\]

By putting all these new conjuncts in \( \varphi \) we clearly obtain a normalized conjunction \( \varphi^* \) in which all the variables of sort \( q > 1 \) are safe. This normalized conjunction is clearly satisfied by the correspondent extension of \( M \). So we can apply the algorithm applied for weak safe formulae and obtain a finite model \( M^* \).

We can then conclude with the following theorem

**THEOREM 5.1** A normalized conjunction of \( \mathcal{L}_p^{2,\infty} \) is weakly satisfiable if and only if it is finitely satisfiable.

\[
\bullet
\]
and the following open problems

(op1) Is the finite satisfiability problem for normalized conjunctions of $L^{2,\infty}$ solvable?

(op2) Is the general satisfiability problem for normalized conjunctions of $L^{2,\infty}$ solvable?

6 Final remarks

Many elementary notions of general topology can be formulated in the basic topological language defined by $2LS$ with singleton and Kuratowski's closure operator, and hence a fortiori in the language $L^{2,\infty}_P$. Among these, for instance, are the boundary and the interior operators, as well as the predicates open(·) and closed(·). Also, the notions of open and closed domains, dense, co-dense, and nowhere dense sets are easily characterizable by $L^{2,\infty}_P$-formulae (see [6]).

The presence of the singleton operator makes it possible to express the notions of accumulation point and, dually, isolated point. More specifically, the predicate $x$ is an accumulation point of $X$ can be expressed by $x \in X \setminus \{x\}$. Dually, $x$ is an isolated point of $X$ is logically equivalent to $x \in X \land x \notin X \setminus \{x\}$.

Let us now show that our satisfiability test is able to prove that validity of the topological statement

if $X = Y \otimes Z$ and $y$ and $z$ are accumulation points of $Y$ and $Z$ respectively then the ordered pair $x = (y, z)$ is an accumulation point for $X$.

The corresponding $L^{2,\infty}_P$ formula is

$$X = Y \otimes Z \setminus X_1 = \{x\}$$

$$\land X_1 = Y_1 \otimes Z_1 \land Y_1 = \{y\} \land Z_1 = \{z\}$$

$$\land Y_2 = Y \setminus Y_1 \land Z_2 = Z \setminus Z_1 \land Y_3 = \overline{Y_2} \land Z_3 = \overline{Z_2}$$

$$\land y \in Y_3 \land z \in Z_3 \land X_2 = X \setminus X_1 \land X_3 = \overline{X_2} \land x \notin X_3$$

It is quite easy to check that all variables are safe.

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References


