The Degrees of Discontinuity of some Translators between Representations of the Real Numbers

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Abstract

Representations like decimal representation are used for defining computability on the set of real numbers. Translatability between different representations has been studied in the past by several authors. Most of the not computably solvable translation problems are not even continuously solvable. In this paper the degrees of discontinuity of translations between a number of common representations are compared and characterized. Mainly three degrees are considered: the first one with translations between the standard representation and the weak cut representations, the second one contains among others the translations between *m*-adic and *n*-adic representations, and the third one contains translations concerning proper cut representations and the iterated fraction representation.

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1. Introduction

Almost every approach to introduce effectivity (constructivity, computability, computational complexity) on the real numbers is (at least indirectly) based on some representation $\delta :\subseteq \Sigma^\omega \longrightarrow \mathbb{R}$, where Σ is a finite alphabet and Σ^ω is the set of all infinite sequences of elements of Σ. If $\delta(p) = x$ then the sequence $p = p(0)p(1)\ldots$ is considered as a name of the real number x. A well-known example is the decimal representation δ_{10}, e.g. $\delta_{10}(3, 14159\ldots) = \pi \in \mathbb{R}$. Representations can be compared by reducibility ("translatability"): δ is computationally reducible to δ', $\delta \leq_c \delta'$, iff there is some computable function $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$, with $\delta(p) = \delta' f(p)$ for all $p \in dom(\delta)$. The representations δ und δ' are called computationally equivalent, $\delta \equiv_c \delta'$, iff $\delta \leq_c \delta'$ and $\delta' \leq \delta$.

The representations on which all the approaches to effectivity in analysis are based are elements of a single equivalence class. It is remarkable that the decimal representation is not in this class. Computational reducibility for a variety of representations if $\mathbb{R}$ has been studied by T. Deil [2].

The Type 2 Theory of Effectivity (TTE) (see e.g. Kreitz and Weihrauch [3], Weihrauch [10] [12]) gave new insights. Computable functions $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ are continuous (w.r.t. to the Cantor topology on Σ^ω). This continuity can be interpreted as a very basic kind of constructivity: (a finite portion of the output depends only on a finite portion of the input).

For representations define: δ is (continuously) reducible to δ', $\delta \leq \delta'$, iff there is some continuous $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ with $\delta(p) = \delta' f(p)$ for all $p \in dom(\delta)$. It turns out that for all representations δ, δ' considered by Deil [2] either $\delta \leq_c \delta'$ or $\delta \not\leq \delta'$ (while Deil has only proved $\delta \not\leq_c \delta'$). This means that in these cases non-translatability is independent of Church's Theses and depends on a more elementary effectivity concept. Similarly, almost all "effectively unsolvable" problems in Analysis turn out to be not continuously solvable in their TTE-formulation.

In ordinary recursion theory, "computable" reducibilities and corresponding degrees of unsolvabiliy are introduced for comparing and classifying the kinds of non–effectivity of sets and functions. Correspondingly, in TTE continuous reducibilities can be used for comparing and classifying kinds of non–continuity (i.e. non–constructivity) of problems in Analysis (v. Stein [7], Mylatz [5]). While for subsets of Σ^ω a nice reducibility theory is avaible (Wadge [8], see also Weihrauch [11] and Staiger/Weihrauch [6]), for functions $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ we have a natural reducibility but almost no knowledge about its structure. Before developing a theory also for this case, more non trivial examples should be studied in detail.

In this paper we introduce a concept of continuous reducibility for "problems" and characterize the degrees of discontinuity for the translation problems between several representations of the real numbers.

The non–continuous translation problems between the standard representation, the left–cut and right–cut r.e. representations and the naive Cauchy representation belong to a single degree. This degree contains the translation from the enumeration representation to the characteristic function representation of the natural numbers.

The m–adic representations are special cases of the "weak separation" representations. It is shown that the non–continuous translation problems between most pairs of weak separation representations (including the m–adic representations) also belong to a single degree of discontinuity which is characterized by a special separation problem of open sets.

Finally, the non–continuous translation problems concerning right–cut, left–cut und continued fraction representations belong to a single degree which can also be characterized by a simple abstract problem.

2. Prerequisites and Basic Definitions

Let Σ be some sufficiently large finite alphabet with $\{0,1\} \subseteq \Sigma$. By Σ^* we denote the finite sequences (words), by Σ^ω the infinite sequences over Σ. Concatenation of words $x, y \in \Sigma^*$ and sequences $p \in \Sigma^\omega$ is denoted by $xy \in \Sigma^*$ and $xp \in \Sigma^\omega$, resp. For $X, Y \subseteq \Sigma^*$, and $T \subseteq \Sigma^\omega$, define $XT := \{xt \mid x \in X, t \in T\}$ (correspondingly for $xY, XY, \ldots$). On Σ^* we consider the discrete topology (every $X \subseteq \Sigma^*$ is open). On Σ^ω we consider the "Cantor topology" τ_C defined by the following basis of open sets:

$$\{w\Sigma^\omega \mid w \in \Sigma^*\}.$$

The standard pairing $<,>: \Sigma^\omega \times \Sigma^\omega \longrightarrow \Sigma^\omega$ is defined by $< p, q >= (p(0), q(0), p(1), q(1), \ldots)$. Tuplings with more than two arguments are defined accordingly. Tuplings and their inverses are continuous. The standard (Cantor) pairing $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ is also denoted by $<,>$ (accordingly for tuplings with more than two arguments).

A representation of a set M is a possibly partial, surjective function

$$\delta :\subseteq \Sigma^\omega \longrightarrow M.$$

The following representations will be used repeatedly (see Weihrauch [10]):

$$\begin{array}{lll}
En & : & \Sigma^\omega \longrightarrow 2^\omega \quad \text{(enumeration representation)} \\
 & & En(p) := \{n \mid 01^{n+1}0 \text{ is a substring of } p\}, \\
Cf & : & \Sigma^\omega \longrightarrow 2^\omega \quad \text{(characteristic function representation)} \\
 & & Cf(p) := \{n \mid p(n) = 1\}, \\
\iota & :\subseteq & \Sigma^\omega \longrightarrow \omega^\omega \quad (\overline{p} := \iota(p)) \\
 & & \iota^{-1}(n_0 n_1 \ldots) := 1^{n_0} 0 1^{n_1} 0 \ldots
\end{array}$$

For $p \in \Sigma^\omega$ let $p[n] := p(0) \ldots p(n-1) \in \Sigma^*$, and for $w \in \Sigma^*$ let $En(w) := \{n \mid 01^{n+1}0 \text{ is a subword of } w\}$.

Continuous reducibility (reducibility for short) for representations is defined by

$$\delta \leq \delta' :\Longleftrightarrow \exists f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega \text{ continuous: } \forall p \in dom(\delta) : \delta(p) = \delta' f(p),$$

$\delta < \delta'$ means $\delta \leq \delta'$ and not $\delta' \leq \delta$. Let

$$(\delta \to \delta') = \{f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega \mid \forall p \in dom(\delta) : \delta(p) = \delta' f(p)\}$$

be the set of all (not necessarily continuous) translators form δ to δ'. For functions $f, g :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ (or Σ^*) define reducibility by

$$f \leq g :\Longleftrightarrow \exists \text{ continuous functions } A, B : \forall p \in dom(f) : f(p) = A(p, gB(p)).$$

A set X of functions $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ (or Σ^*) may be considered as the set of solutions of a problem P_X (e.g. the set $(\delta \to \delta')$ is the set of solutions of the problem to translate δ to δ'). We compare the "difficulty" of such problems by a uniform reducibility. For sets X, Y of functions $:\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ (or Σ^*) define:

$$\begin{array}{ll} X \leq Y :\Longleftrightarrow & \text{there are continuous functions } A, B : \\ & \forall g \in Y : \exists f \in X : \forall p \in dom(f) : f(p) = A(p, gB(p)) \end{array}$$

Thus A and B define a uniform method which for every solution g of Problem P_Y produces a solution of Problem P_X. Clearly $f \leq g \Longleftrightarrow \{f\} \leq \{g\}$, hence $f \leq g$ may be considered as a short notation of $\{f\} \leq \{g\}$. The relation $\leq$ is transitive. As usually, define $X \equiv Y :=\Longleftrightarrow X \leq Y$ and $Y \leq X$ (equivalence). The equivalence classes can be called "discontinuity degrees of problems".

3. Representations Below the Standard Representation

Let $Q \subseteq \mathbb{R}$ be some dense subset of $\mathbb{R}$, and let $\nu_Q : \mathbb{N} \longrightarrow Q$ be some (total) numbering of Q. We define four representations $:\subseteq \Sigma^\omega \longrightarrow \mathbb{R}$ of the reals numbers.

Definition 1

(1) Naive Cauchy representation δ_C:

$$\delta_C(p) = x \iff \text{the sequence } \nu_Q\overline{p}(0), \nu_Q\overline{p}(1), \ldots \text{ converges to } x$$

for all $p \in \Sigma^\omega$ and $x \in \mathbb{R}$.

(2) R.e. left cut representation $\delta_<$:

$$\delta_<(p) = x :\iff En(p) = \{i \mid \nu_Q(i) < x\}$$

for all $p \in \Sigma^\omega$ and $x \in \mathbb{R}$.

(3) R.e. right cut representation $\delta_>$:

$$\delta_>(p) = x :\iff En(p) = \{i \mid \nu_Q(i) > x\}$$

for all $p \in \Sigma^\omega$ and $x \in \mathbb{R}$.

(4) Standard representation δ_{st}:

$$\delta_{st} < p, q >= x :\iff \delta_<(p) = \delta_>(q) = x$$

for all $p, q \in \Sigma^\omega$ and $x \in \mathbb{R}$.

For the special case that ν_Q is a standard numbering of the rational numbers, e.g. $\nu < i, j, k >:= (i - j) \setminus (1 + k)$, translatability has been discussed in detail by Deil [2] (computational) and Weihrauch [10] (continuous). For this ν_Q, the representation δ_{st} is computationally equivalent to those representations of the real numbers on which almost all approaches to effective analysis are based.
The degrees of δ_C, $\delta_<$, $\delta_>$ and δ_{st}, do not depend on the particular choice of Q und ν_Q.

Lemma 2
Let R be dense in $\mathbb{R}$ and let $\nu_R : \mathbb{N} \longrightarrow R$ be a numbering of R. Define δ'_C, $\delta'_<$, $\delta'_>$ and δ'_{st} by replacing ν_R for ν_Q in Definition 1. Then

$$\delta'_C \equiv \delta'_C, \;\; \delta'_< \equiv \delta'_<, \;\; \delta'_> \equiv \delta'_>, \;\; \delta'_{st} \equiv \delta'_{st}.$$

Proof:

$\delta_C \leq \delta'_C$: There is a function $h : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ with $|\nu_Q(i) - \nu_R h(i,j)| < 2^{-j}$. Define a functionn $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ by $dom(f) := dom(\iota)$ and

$$f(1^{n_0}01^{n_1}0\ldots) := 1^{h(n_0)}01^{h(n_1,1)}0\ldots$$

Then f is continuous and $\delta_C(p) = \delta'_C f(p)$ for all $p \in dom(\delta_C)$.

$\delta_< \leq \delta'_<$: For given $p \in \Sigma^\omega$ define words $w_i \in \Sigma^*$ $(i = 0, 1, \ldots)$ by

$$w_{<i,k,m>} := \begin{cases} 01^{m+1}0 & \text{if } \nu_R(m) < \nu_Q(k) \text{ and } 01^{k+1}0 \text{ is a} \\ & \text{subword of } p(0)\ldots p(i) \\ 0 & \text{otherwise} \end{cases}$$

and set $f(p) := w_0 w_1 \ldots$ Then f is continuous and

$$m \in En\, f(p) \Longleftrightarrow \exists k \in En(p) : \nu_R(m) < \nu_Q(k),$$

hence $\delta_<(p) = \delta'_< f(p)$.
The other statements are proved correspondingly.
Q.E.D.

The following translatability results are well known [2, 10]:

Theorem 3

(1) $\delta_{st} < \delta_< < \delta_C, \;\; \delta_{st} < \delta_> < \delta_C,$

(2) $\delta_< \not\leq \delta_>$ and $\delta >\not\leq \delta_<$.

(3) δ_{st} is (up to equivalence) the least upper bound of $\delta_<$ and $\delta_>$

The proofs are easy. For illustration two examples are included.

Proof

$\delta_< \leq \delta_C$: For $p \in \Sigma^\omega$ define a sequence $w_0, w_1, \ldots$ of words as follows:

$$w_0 := \varepsilon$$
$$w_n := \begin{cases} 0 & \text{if } En(p[n]) = \emptyset \\ 01^{k+1}0 & \text{if } k \in En(p[n]) \text{ and} \\ & \nu_Q(m) \leq \nu_Q(k) \text{ for all } m \in En(p[n]). \end{cases}$$

Define $f(p) := w_0 w_1 \ldots$. Then $f : \Sigma^\omega \longrightarrow \Sigma^\omega$ is continuous and $\delta_<(p) = \delta_C f(n)$ for all $p \in dom(\delta_<)$.

$\delta_< \not\leq \delta_>$: Assume, $\delta_< \leq \delta_>$. Then there is some continuous $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ with $\delta_<(p) = \delta_> f(p)$ for all $p \in dom(\delta_<)$. Let $p = 01^{n_0}01^{n_1}0\ldots$ with $\delta_<(p) = x$. Then $\delta_> f(p) = x$. Hence for some k, n $01^{k+1}0$ is a suffix of $f(p)[n]$. Consequently, $x < \nu_Q(k)$. By continuity of f, there is a prefix v of p such that $f(v\Sigma^\omega) \subseteq f(p)[n]\Sigma^\omega$. There is some $r \in \Sigma^\omega$ such that $\delta_<(vr) > \nu_Q(k)$, but $f(vr) \in f(p)[n]\Sigma^\omega$, hence $\nu_Q(k) > \delta_> f(vr) = x$ (contradiction).
Q.E.D.

By the next theorem the essential parts of which are from v. Stein [7] all the not continuously solvable translation problems between the above representations are in the same degree of discontinuity, namely in the degree of $EC : \Sigma^\omega \longrightarrow \Sigma^\omega$, the translator from the enumeration representation En to the characteristic function representation Cf of 2^ω (i.e. $\{EC\} = (En \longrightarrow Cf)$). Notice that EC is a "very non continuous" function.

Theorem 4

The following sets of translations are equivalent:
$(En \longrightarrow Cf)$, $(\delta_C \longrightarrow \delta_<)$, $(\delta_C \longrightarrow \delta_>)$, $(\delta_C \longrightarrow \delta_{st})$, $(\delta_< \longrightarrow \delta_>)$, $(\delta_> \longrightarrow \delta_<)$, $(\delta_< \longrightarrow \delta_{st})$, $(\delta_> \longrightarrow \delta_{st})$.

Proof

Proposition 1: $\{EC\} \leq (\delta_< \longrightarrow \delta_>)$, $\{EC\} \leq (\delta_> \longrightarrow \delta_<)$.
Proof 1: For $p \in \Sigma^\omega$ define a sequence $w_0, w_1, \ldots$ of words by

$$w_{<k,m>} := \begin{cases} 01^{m+1}0 & \text{if } \nu_Q(m) < \sum\{3^{-i} \mid i \in En(p[k])\} \\ 0 & \text{otherwise} \end{cases}$$

and define $f(p) := w_0, w_1 \ldots$. Then f is continuous, and

$$\delta_< f(p) = \sum\{3^{-i} \mid i \in En(p)\}$$

for all $p \in \Sigma^\omega$. Define a function $g : \Sigma^\omega \times \Sigma^\omega \longrightarrow \Sigma^\omega$ als follows.
For arguments $r, s \in \Sigma^\omega$ define a sequence c_k by the following recursion.

$k = 0$:
Choose some m_0 with $1/2 < \nu_Q(m_0) < 1$. Define

$$j_0 := \mu j[m_0 \in En(r[j]) \text{ or } m_0 \in En(s[j])]$$

$$c_0 := \begin{cases} div & \text{if } j_0 \text{ does not exists} \\ 1 & \text{if } m_0 \in En(r[j_0]) \\ 0 & \text{otherwise.} \end{cases}$$

$k - 1 \rightarrow k$:
If c_{k-1} does not exist then c_k does not exist. Assume $c_0, \ldots, c_{k-1}$ have been determined. Define

$$a_k := \sum\{3^{-i} \mid i < k, c(i) = 1\}$$

and choose some m_k with

$$a_k + 3^{-k}/2 < \nu_Q(m_k) < a_k + 3^{-k}.$$

Define

$$j_k := \mu j[m_k \in En(r[j]) \text{ or } m_k \in En(s[j])]$$

$$c_k := \begin{cases} div & \text{if } j_k \text{ does not exists} \\ 1 & \text{if } m_k \in En(r[j_k]) \\ 0 & \text{otherwise.} \end{cases}$$

Define $g(r,s) = c_0c_1\ldots$ if c_i exists for all i, $g(r,s) = div$ otherwise. Then g is continuous.

Proposition 2: Assume $\delta_<(r) = \delta_>(s) = \sum\{3^{-1} \mid i \in En(p)\} =: x_p$. Then for all k $g(r,s)(k)$ exists and $k \in En(p) \Longleftrightarrow g(r,s)(k) = 1$.

Proof 2: By assumption $En(r) \cap En(s) = \emptyset$. Let $m_0, m_1, \ldots$ be the numbers of the above construction for r and s.

$k = 0$:

$0 \in En(p) \Longrightarrow x_p \geq 1 \Longrightarrow \nu_Q(m_0) < x_p \Longrightarrow m_0 \in En(r) \setminus En(s) \Longrightarrow g(r,s)(0) = 1$

$0 \notin En(p) \Longrightarrow x_p \leq 1/2 \Longrightarrow \nu_Q(m_0) > x_p \Longrightarrow m_0 \in En(s) \setminus En(r) \Longrightarrow g(r,s)(0) = 0$

$k - 1 \to k$:

By induction, $c_0, \ldots, c_{k-1}$ exists. We obtain

$$\begin{aligned} k \in En(p) &\Longrightarrow x_p \geq a_k + 3^{-k} \Longrightarrow \nu_Q(m_k) < x_p \Longrightarrow m_k \in En(r) \setminus En(s) \\ &\Longrightarrow g(r,s)(k) = 1 \\ k \notin En(p) &\Longrightarrow g(r,s)(k) = 0 \text{ (accordingly)} \end{aligned}$$

q.e.d. (2)

Define $h(p,q) := g(f(p),q)$. Then h is continuous.
Let $T \in (\delta_< \longrightarrow \delta_>)$. Then for any $p \in \Sigma^\omega$, $\delta_< f(p) = \delta_> Tf(p) = \sum\{3^{-i} \mid i \in En(p)\}$, hence by Proposition 2, $g(f(p), Tf(p))$ exists and $\forall k : k \in En(p) \Longleftrightarrow g(f(p), Tf(p))(k) = 1 \Longleftrightarrow h(p, Tf(p))(k) = 1$. Therefore, for all $T \in (\delta_< \longrightarrow \delta_>)$ $EC(p) = h(p, Tf(p))$, hence $\{EC\} \leq (\delta_< \longrightarrow \delta_>)$. $\{EC\} \leq (\delta_> \longrightarrow \delta_<)$ can be proved accordingly.
q.e.d. (1)

Proposition 3: $(\delta_C \longrightarrow \delta_{st}) \leq \{EC\}$.

Proof 3: For $p \in \Sigma^\omega$ define a sequence $w_0, w_1, \ldots$ of words by:

$$w_{<i,n,k>} := \begin{cases} 01^{1+<n,k>}0 & \text{if } i > k \text{ and } |\nu_Q\overline{p}(k) - \nu_Q\overline{p}(i)| \geq 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

and set $f(p) := w_0 w_1 \ldots$ Then f is continuous and

$$En\, f(p) = \{< n, k > \mid \exists i > k : |\nu_Q\overline{p}(k) - \nu_Q\overline{p}(i)| \geq 2^{-n}\},$$

therefore,

$$EC\, f(p) < n, k > \neq 1 \iff (\forall i > k)|\nu_Q\overline{p}(k) - \nu_Q\overline{p}(i)| < 2^{-n}.$$

For $p, r \in \Sigma^\omega$ define sequences x_i, y_i ($i \in \mathbb{N}$) of words by

$$x_{<m,k,n>} := \begin{cases} 01^{m+1}0 & \text{if } r < n, k > \neq 1 \text{ and } \nu_Q(m) < \nu_Q\overline{p}(k) - 2^{-n} \\ 0 & \text{otherwise,} \end{cases}$$

$$y_{<m,k,n>} := \begin{cases} 01^{m+1}0 & \text{if } r < n, k > \neq 1 \text{ and } \nu_Q(m) > \nu_Q\overline{p}(k) + 2^{-n} \\ 0 & \text{otherwise,} \end{cases}$$

and define $g_1(p,r) := x_0 x_1 \ldots$ and $g_2(p,r) := y_0 y_1 \ldots$
Then g_1 and g_2 be continuous. If $\delta_C(p) = x$ and $r = EC\, f(p)$ then $En(g_1(p,r)) = \{m \mid \nu_Q(m) < x\}$ and $En(g_2(p,r)) = \{m \mid \nu_Q(m) > x\}$. Define h by $h(p,r) :=< (g_1(p,r), g_2(p,r) >$. Then h is continuous and the function t with $t(p) = h(p, EC\, f(p))$ translates δ_C into δ_{st}.
q.e.d. (3)

From Theorem 3 and Proposition 1 und 3 we obtain

$$\{EC\} \leq \left\{ \begin{array}{l} (\delta_< \to \delta_>) \leq \left\{ \begin{array}{l} (\delta_C \to \delta_>) \\ (\delta_< \to \delta_{st}) \end{array} \right. \\ \\ (\delta_> \to \delta_<) \leq \left\{ \begin{array}{l} (\delta_C \to \delta_<) \\ (\delta_> \to \delta_{st}) \end{array} \right. \end{array} \right\} \leq (\delta_C \to \delta_{st}) \leq \{EC\}$$

This proves our theorem.
Q.E.D.

4 The class of m–adic and Related Representations

Reducibility between m–adic representations has been investigated in the past by may authors (see Deil [2]).
Here we shall charaterize the degree of the non continuously solvable translation problems.

Definition 5 *(m–adic representation)*

Let Σ be a finite alphabet and $m \in \mathbb{N}$, $m \geq 2$ with $\{0, 1, \ldots, m-1, +1, -1, \bullet\} \subseteq \Sigma$. the m–adic representation $\delta_m :\subseteq \Sigma^\omega \longrightarrow \mathbb{R}$ of $\mathbb{R}$ is defined by

$$\delta_m(sa_n \ldots a_0 \bullet a_{-1}a_{-2} \ldots) = s \cdot \sum\{a_i m^i \mid i \leq n\}$$

where $s \in \{+1, -1\}$ and $a_i \in \{0, \ldots, m-1\}$ for $i \leq n$. $\delta_m(p)$ is undefined for arguments p which do not have this form.

First, we embed the m–adic representation into a more general class.

Definition 6 *(weak separation representation)*

Let $Q \subseteq \mathbb{R}$ be dense in $\mathbb{R}$ and let $\nu : \mathbb{N} \longrightarrow Q$ be a numbering of Q. Define $\delta_\nu :\subseteq \Sigma^\omega \longrightarrow \mathbb{R}$ by

$$\delta_\nu(r) = x :\Longleftrightarrow \forall i \begin{cases} r(i) = 0 & \text{if} \quad \nu_Q(i) < x \\ & \text{and} \\ r(i) \neq 0 & \text{if} \quad \nu_Q(i) > x \end{cases}$$

for all $r \in \Sigma^\omega$ and $x \in \mathbb{R}$.

Notice that $r(i)$ is arbitary if $\nu(i) = x$.
Reducibility between weak separation representations can be characterized easily.

Theorem 7

Let Q, R be dense in $\mathbb{R}$, let $\nu : \mathbb{N} \longrightarrow Q$, $\mu : \mathbb{N} \longrightarrow R$, be numberings and let δ_ν and δ_μ be the corresponding weak separation representations. Let δ_{st} be the standard representation (Def.1). Then:

(1) $\delta_\nu < \delta_{st}$,

(2) $R \subseteq Q \Longleftrightarrow \delta_\nu \leq \delta_\mu$.

Proof

First we prove (2).
Consider the case $R \subseteq Q$. Then there is some function $h : \mathbb{N} \longrightarrow \mathbb{N}$ with $\mu(i) = \nu h(i)$ for all i. Define $f : \Sigma^\omega \longrightarrow \Sigma^\omega$ by $f(r)(i) := rh(i)$ for all $r \in \Sigma^\omega$ and $i \in \mathbb{N}$. Then f is continuous. If δ_ν(r)=x we obtain for all i:

$$\begin{array}{llll} \mu(i) < x & \Longrightarrow \nu h(i) < x & \Longrightarrow rh(i) = 0 & \Longrightarrow f(r)(i) = 0, \\ \mu(i) > x & \Longrightarrow \nu h(i) > x & \Longrightarrow rh(i) \neq 0 & \Longrightarrow f(r)(i) \neq 0, \end{array}$$

hence $\delta_\nu(r) = x = \delta_\mu f(r)$. This proves $\delta_\nu \leq \delta_\mu$.

If especially $R = Q$ then for any two numerings ν und μ of Q we have $\delta_\nu \equiv \delta_\mu$.

Consider now the case $R \not\subseteq Q$. Let us assume first that $\nu : \mathbb{N} \longrightarrow Q$ and $\mu : \mathbb{N} \longrightarrow R$ are bijective. By assumption there is some k with $x := \mu(k) \in R \setminus Q$.

Since $x \notin Q$, for all i either $\nu(i) < x$ or $\nu(i) > x$, hence by injectivity of ν there is only one $p \in \Sigma^\omega$ with $\delta_\nu(p) = x$. If $\delta_\mu(q) = x$, then $q(i)$ is uniquely defined for all $i \neq k$ since $x = \mu(k)$ und μ is injective. Now let $T :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ be translator from δ_Q to δ_R, i.e. $T \in (\delta_Q \longrightarrow \delta_R)$. Assume that T is continuous. We have $x = \delta_\nu(p) = \delta_\mu T(p)$.
Case $T(p)(k) = 0$:
Let $v \in \Sigma^\omega$ be the prefix of $T(p)$ of legth k. Then $T(p) \in v0\Sigma^\omega$. By continuity of T there is some prefix u of p such that $T(u\Sigma^\omega) \subseteq v0\Sigma^\omega$.

Let $n := length\,(u)$. Since $\nu(i) \neq x$ for all i, we have

$$a := max\,\{\nu(i) \mid i < n, p(i) = 0\} < x < min\,\{\nu(i) \mid i < n, p(i) \neq 0\} =: b$$

There is some $y \in \mathbb{R}$ with $a < y < x$. Let $p' \in \Sigma^\omega$ with $\delta_\nu(p') = y$. Since $a < y < x$, obviously $p' \in u\Sigma^\omega$, hence $T(p') \in v0\Sigma^\omega$ and $T(p')(k) = 0$. On the other hand $\delta_\mu T(p') = y < x = \mu(k)$, hence $T(p')(k) \neq 0$ by definition of δ_μ (contradiction).

Case $T(p)(k) \neq 0$:
A contradiction can be derived accordingly. Therefore, there is no continuous translator T, i.e. $\delta_\nu \not\leq \delta_\mu$.

Now let $\nu : \mathbb{N} \longrightarrow Q$ and $\mu : \mathbb{N} \longrightarrow R$ be arbitrary and $R \subseteq Q$. There are bijective numberings $\nu' : \mathbb{N} \longrightarrow Q$ and $\mu' : \mathbb{N} \longrightarrow R$. From the first part of our proof we know $\delta_\nu \equiv \delta_{\nu'}$ and $\delta_\mu \equiv \delta_{\mu'}$, from the second part $\delta_{\nu'} \not\leq \delta_{\mu'}$, hence $\delta_\nu \not\leq \delta_\mu$.

Finally, we prove (1).
For any $p \in \Sigma^\omega$ define words $w_0, w_1, \ldots \in \Sigma^*$ by

$$w_{<i,j>} := \begin{cases} 01^{j+1}0 & \text{if } \nu(j) < \nu(i) \text{ and } p(i) = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and define $f_<(p) = w_0 w_1 \ldots$. Then $f_<$ is continuous and $\delta_\nu(p) = \delta_< f_<(p)$ for all $p \in dom\,(\delta_Q)$. Correspondingly there is a continuous translation $f_>$ from δ_ν to $\delta_>$. Define $f(p) :=< f_<(p), f_>(p) >$. Then $f \in (\delta_\nu \longrightarrow \delta_{st})$. This shows $\delta_\nu \leq \delta_{st}$. Assume $\delta_{st} \leq \delta_\nu$. Then for all numberings $\mu : \mathbb{N} \longrightarrow R$, R dense in $\mathbb{R}$: $\delta_\mu \leq \delta_{st} \leq \delta_\nu$ which is false by (2) of this theorem.
Q.E.D.

Since the equivalence class of δ_ν does not depend on the particular numbering $\nu : \mathbb{N} \longrightarrow Q$ of Q, we shall use the notation δ_Q for δ_ν considering some fixed numbering implicitly.

There are several interesting representations which are equivalent to weak separation representations (Deil [2]). We prove this only for the m-adic representations.

Theorem 8

Let δ_m be the m-adic representation and let $\nu < i, j, k > := (i-j)/m^k$ and $Q := range\,(\nu)$. Then

$$\delta_Q \equiv \delta_m.$$

Proof:

$\delta_m \leq \delta_Q$:
For $p = +1a_n \ldots a_0 \bullet a_{-1} \ldots$ define

$$\Gamma_+(p) < i, j, k > := \begin{cases} 0 & \text{if } \nu < i, j, k > \leq a(p, k) \\ 1 & \text{if } \nu < i, j, k > > a(p, k) \end{cases}$$

where $a(p, k) := (a_n \ldots a_0 \bullet a_{-1} \ldots a_{-k})_m := \sum\{a_i m^i \mid n \geq i \geq -k\}$. Then Γ_+ is continuous on $+1\Sigma^\omega \cap dom\,(\delta_m)$. For all $< i, j, k >$ and all $p \in +1\Sigma^\omega \cap dom\,(\delta_m)$ we have

$$\begin{array}{llll} \nu < i, j, k > < \delta_m(p) & \Longrightarrow \nu < i, j, k > \leq a(p, k) & \Longrightarrow \Gamma_+(p) < i, j, k > & = 0 \\ \nu < i, j, k > > \delta_m(p) & \Longrightarrow \nu < i, j, k > > a(p, k) & \Longrightarrow \Gamma_+(p) < i, j, k > & \neq 0 \end{array}$$

Therefore $\delta_Q \Gamma_+(p) = \delta_m(p)$. Accordingly there is a continuous function Γ_- with $\delta_Q \Gamma_-(p) = \delta_m(p)$ for all $p \in -1\Sigma^\omega \cap dom\,(\delta_m)$. Therefore Γ with $\Gamma(p) = \Gamma_+(p)$ if $p(0) = +1$, $\Gamma_-(p)$ otherwise) is a continuous translator from δ_m to δ_Q.

$\delta_Q \leq \delta_m$:
If $\delta_Q(p) = x$ and $p(0) = 0$ then $\nu(0) \leq x$, hence $0 = \nu < 0, 0, 0 >= \nu(0) \leq x$. First, for a given $p \in dom\,(\delta_Q)$ with $p(0) = 0$ we determine n and $a_n, a_{n-1,\ldots}$ such that $\delta_Q(p) = \delta_m(+1a_n a_{n-1} \ldots a_0 \bullet a_{-1} \ldots)$.

Step 0: Let i_0 be the number with $p < i_0, 0, 0 >= 0$ and $p < i_0+1, 0, 0 > \neq 0$. Then

$$i_0 = \nu < i_0, 0, 0 > \leq \delta_Q(p) \leq \nu < i_0 + 1, 0, 0 >= i_0 + 1.$$

Determine n and $a_n, \ldots, a_0 \in \{0, \ldots, m-1\}$ such that $i_0 = (a_n \ldots a_0)_m$.

Step $k+1$: Assume that $a_n, \ldots, a_{-k}$ and b_k and c_k have been determined with

$$\nu(b_k) = (a_m \ldots a_0 \,.\, a_1 \ldots a_{-k})_m \leq \delta_Q(p) \leq (a_m \ldots a_0 \,.\, a_{-1} \ldots a_{-k})_m + m^{-k} = \nu(c_k)$$

and $p(b_k) = 0$ and $p(c_k) \neq \emptyset$. Define $d_0 := b_k$, $d_m := c_k$, and for $j = 1, \ldots, m-1$ choose d_j such that $\nu(d_j) = \nu(d_0) + j \cdot m^{-(k+1)}$. Let $a_{-(k+1)}$ be the greatest j with $p(d_j) = 0$ and let $b_{k+1} := d_j$ and $c_{k+1} := d_{j+1}$ for this j. Then

$$\begin{aligned} \nu(b_{k+1}) = \ & (a_m \ldots a_0 \,.\, a_{-1} \ldots a_{-(k+1)})_m \leq \\ & \delta_Q(p) \leq (a_m \ldots a_0 \,.\, a_{-1} \ldots a_{-(k+1)})_m + m^{-(k+1)} = \nu(c_{k+1}) \end{aligned}$$

and $p(b_{k+1}) = 0$ and $p(c_{k+1}) = 1$.

By this procedure a continuous function $f_+ :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ is defined such that for all $p \in 0\Sigma^\omega \cap dom\,(\delta_Q)$: $\delta_Q(p) = \delta_m f_+(p)$. Accordingly, there is a continuous funktion f_- such that $\delta_Q(p) = \delta_m f_-(p)$ if $p \in dom\,(\delta_Q) \setminus 0\Sigma^\omega$. A combination of f_+ and f_- gives a continuous translator from δ_Q to δ_m.
Q.E.D.

As a corollary of Theorem 7 and 8 we obtain the well known translatability result for m–adic representations [2, 10].

Corollary 9

For all $m, n \geq 2$:

(1) $\delta_m < \delta_{st}$,

(2) n divedes a power of $m \Longleftrightarrow \delta_m \leq \delta_n$.

Proof:

(1) $\delta_m \equiv \delta_Q < \delta_{st}$

(2) Let $Q_m := \{(i-j)/m^k \mid i,j,k \in \mathbb{N}\}$, $Q_n = \{(i-j)/n^k \mid i,j,k \in \mathbb{N}\}$.

Then $\delta_m \leq \delta_n \iff \delta_{Q_m} \leq \delta_{Q_n} \iff Q_n \subseteq Q_m \iff n$ divides a power of m.
Q.E.D.

Our next aim is to characterize degrees of translators between weak separation representations. For this purpose we introduce a problem which is not formulated in terms of the reals numbers.

Definition 10 *(1–separation problem)*

Let SEP1 be the set of all functions $f :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ such that for all $p, q \in \Sigma^\omega$ with $En\,(p) \cap En\,(q) = \emptyset$ and $card\,(\mathbb{N} \setminus (En\,(p) \cup En(q)) \leq 1$ $f < p, q >$ exists and

$$\forall i : f < p, q > (i) = \begin{cases} 0 & \text{if } i \in En(p) \\ 1 & \text{if } i \in En(q). \end{cases}$$

Thus the set $X := (f < p, q >)^{-1}\{0\}$ separates $En(p)$ and $En(q)$, since $En(p) \subseteq X \subseteq \mathbb{N} \setminus En(q)$. We shall call SEP1 the 1–separation problem. Notice the formal similarity between the 1–separation problem and the problem of finding δ_Q–names.

Lemma 11

Let Q be dense in $\mathbb{R}$, let $\nu : \mathbb{N} \longrightarrow Q$ be a bijective numbering and let δ_{st} be the derived standard representation (Def.1) and δ_Q the derived weak separation representation. Then

$$(\delta_{st} \longrightarrow \delta_Q) \leq SEP\,1$$

Proof

Let $S \in SEP\,1$ be a separator. Assume $< p, q > \in dom\,(\delta_{st})$, $x := \delta_{st} < p, q >$. Then $En(p) \cap En(q) = \emptyset$, $En(p) \cup En(q) = \mathbb{N}$ if

$\delta_{st} < p,q > \notin Q$, $En(p) \cup En(q) = \mathbb{N} \setminus \{k\}$ for the k with $\nu_Q(k) = \delta_{st} < p,q >$ otherwise. Remember that ν is injective. Therefore, $< p,q > \in dom\,(S)$ and for all i:

$$\begin{array}{lll} \nu_Q(i) < x & \Longrightarrow i \in En(p) & \Longrightarrow S < p,q > (i) \ = 0, \\ \nu_Q(i) > x & \Longrightarrow i \in En(q) & \Longrightarrow S < p,q > (i) \ \neq 0. \end{array}$$

This shows that $\delta_Q S < p,q > = \delta_{st} < p,q >$, therefore $S \in (\delta_{st} \longrightarrow \delta_Q)$. We conclude $SEP\,1 \subseteq (\delta_{st} \longrightarrow \delta_Q)$, hence $(\delta_{st} \longrightarrow \delta_Q) \leq SEP\,1$.
Q.E.D.

Lemma 12

Consider P, Q dense in $\mathbb{R}$, $P \cap Q = \emptyset$, $\nu_P : \mathbb{N} \longrightarrow P$, $\nu_Q : \mathbb{N} \longrightarrow Q$ bijective numberings, δ_P and δ_Q the derived weak separation representations of $\mathbb{R}$. Then

$$SEP\,1 \leq (\delta_P \longrightarrow \delta_Q).$$

Proof

For $p \in \Sigma^\omega$, $b \in \mathbb{N}$ let

$$H(p,n) := \begin{cases} \emptyset & \text{if for no } k \quad 01^{k+1}0 \text{ is a suffix of } p[n] \\ k & \text{if} \qquad\qquad 01^{k+1}0 \text{ is a suffix of } p[n]. \end{cases}$$

For $p, q \in \Sigma^\omega$ define $a_i, m_i, b_i, d_i \in \mathbb{N}, X_i \subseteq \mathbb{N}$ and a function $\xi :\subseteq \mathbb{N} \longrightarrow \mathbb{N}$ in steps $i = 0, 1, \ldots$ als follows.
Step 0:

$$m_0 := 0, d_0 := 0, X_0 := \emptyset, \xi(m_0) := 0.$$

Choose a_0, b_0 with

$$\nu_Q(a_0) < \nu_Q \xi(m_0) < \nu_Q(b_0)$$

and

$$\nu_Q(b_0) - \nu_Q(a_0) \leq 1.$$

Step i, $i = 2n+1$:

– Case $H(p,n) = \emptyset$ or $H(p,n) \in X_{i-1}$:

$$m_i = m_{i-1}, d_i := d_{i-1}, X_i := X_{i-1}.$$

Choose a_i, b_i with

$$\nu_Q(a_{i-1}) < \nu_Q(a_i) < \nu_Q\xi(m_i) < \nu_Q(b_i) < \nu_Q(b_{i-1})$$

and

$$\nu_Q(b_i) - \nu_Q(a_i) < 2^{-n}.$$

– Case $H(p,n) = k, k \neq m_{i-1},\ k \notin X_{i-1}$:
Define m_i, d_i, a_i, b_i as in the case above and define additionally

$$X_i := X_{i-1} \cup \{k\}, \xi(k) := a_i$$

– Case $H(p,n) = m_{i-1} \notin X_i$:

$$\begin{aligned} X_i &:= X_{i-1} \cup \{m_{i-1}\}, \\ m_i &:= min\,(\mathbb{N} \setminus X_i) \\ d_i &:= d_{i-1} + 1 \end{aligned}$$

Define $a_i, b_i, \xi(m_i)$ such that

$$\nu_Q\xi(m_{i-1}) < \nu_Q(a_i) < \nu_Q\xi(m_i) < \nu_Q(b_i) < \nu_Q(b_{i-1})$$

and

$$\nu_Q(b_i) - \nu_Q(a_i) < 2^{-n}$$

and

$$\nu_P(d_{i-1}) \notin [\nu_Q(a_i); \nu_Q(b_i)].$$

Step i, $i = 2n+2$:
(accordingly with q instead of i, but to left of $\nu_Q\xi(m_{i-1})$ in the third case)

For all i either $X_{i-1} = X_i$ and $m_{i-1} = m_i$ or $X_i \cup \{m_i\}$ extends $X_{i-1} \cup \{m_{i-1}\}$ by a single $j \in \mathbb{N}$ for which $\xi(j)$ is defined in Step i. Therefore ξ is a well-defined function. By the construction

$$dom\,(\xi) = En(p) \cup En(q) \cup \{min\,(\mathbb{N} \setminus (En(p) \cup En(q))\},$$

and the function $f_1 : \Sigma^\omega \longrightarrow \Sigma^\omega$ defined by

$$f_1 < p, q > (i) = \begin{cases} \xi(i) & \text{if } (\forall i)\xi(i) \text{ exists} \\ div & \text{otherwise} \end{cases}$$

for all $p, q \in \Sigma^\omega$ is continuous. For any p, q the sequence $[\nu_Q(a_i); \nu_Q(b_i)]$ converges to a number $x_{<p,q>} \in \mathbb{R}$. If $En(p) \cup En(q) \neq \mathbb{N}$, then the sequence $m_0, m_1, \ldots$ converges to $k := min\,(\mathbb{N} \setminus (En(p) \cup En(q))$, hence $x_{<p,q>} = \nu_Q(k) \in Q$. If $En(p) \cup En(q) = \mathbb{N}$ then Case 3 occurs for infinitely many i and $\{d_0, d_1, \ldots\} = \mathbb{N}$. For such a step i, $\nu_P(d_{i-1}) \notin [\nu_Q(a_i); \nu_Q(b_i)]$ hence $\nu_P(d_{i-1}) \neq x_{<p,q>}$. Therefore $(\forall j)\nu_P(d_j) \neq x_{<p,q>})$ i.e. $x_{<p,q} \notin P$. Since $P \cap Q = \emptyset$, $x_{<p,q>} \notin P$ in any case.
Proposition: There is a continuous function $h : \Sigma^\omega \longrightarrow \Sigma^\omega$ with

$$x_{<p,q>} = \delta_P\, f_2 < p, q > \text{ for all } p, q \in \Sigma^\omega.$$

Proof: For $p, q \in \Sigma^\omega$ and $k \in \mathbb{N}$ construct $h < p, q > (k)$ als follows. Find the smallest i such that $\nu_P(k) \notin [\nu_Q(a_i); \nu_Q(b_i)]$. Since $x_{<p,q>} \notin P$ such a number i exists. Define

$$h < p, q > (k) = \begin{cases} 0 & \text{if } \nu_P(k) < \nu_Q(a_i) \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, $x_{<p,q>} = \delta_P h < p, q >$. The function h is continuous.
q.e.d.

The construction guarantees that $\nu_Q\xi(k) < x_{<p,q>}$ if $k \in En(p)$ and $\nu_Q\xi(k) > x_{<p,q>}$ if $k \in En(q)$. Now define

$$g(r, t)(k) := t(f_1(r))(k)).$$

Then g is continuous. We shall prove now that for any $T \in (\delta_p \longrightarrow \delta_Q)$ the function S with $S(r) = g(r, Th(r))$ is a 1-separator $S \in SEP\,1$.

Let $r :=< p, q >$ such that $En(p) \cup En(q) = \mathbb{N}$ and $card\,(\mathbb{N} \setminus (En(p) \cup En(q))) \leq 1$. Then $\xi = f_1 < p, q >$ exists and for all $i \in \mathbb{N}$:

$$\begin{aligned} i \in En(p) &\Longrightarrow \nu_Q\xi(i) < x_{<p,q>} \\ &\Longrightarrow \nu_Q f_1(r)(i) < \delta_P h(r) \\ &\Longrightarrow \nu_Q f_1(r)(i) < \delta_Q Th(r) \\ &\Longrightarrow Th(r)(f_1(r)(i)) = 0 \\ &\Longrightarrow g(r, Th(r))(i) = 0 \end{aligned}$$

and $i \in En(q) \Longrightarrow g(r, Th(r))(i) = 0$ (correspondengly). Therefore, S is a 1–separator. This proves $SEP1 \leq (\delta_P \longrightarrow \delta_Q)$
Q.E.D.

We combine the results as follows.

Theorem 13

Let $P, Q \subseteq \mathbb{R}$ such that P and $Q \setminus P$ are dense in $\mathbb{R}$. Then

$$SEP1 \equiv (\delta_{st} \longrightarrow \delta_P) \equiv (\delta_P \longrightarrow \delta_Q) \quad .$$

Proof

$$\begin{array}{rlll} SEP1 & \leq & (\delta_P \longrightarrow \delta_{Q\setminus P}) & \text{(Lemma 12)} \\ & \leq & (\delta_P \longrightarrow \delta_Q) & \text{(since } \delta_Q \leq \delta_{Q\setminus P}\text{, Theorem 7)} \\ & \leq & (\delta_{st} \longrightarrow \delta_Q) & \text{(since } \delta_P \leq \delta_{st}\text{, Theorem 7)} \\ & \leq & SEP1 & \text{(Lemma 11)} \end{array}$$

Q.E.D.

Theorem 13 characterizes the degree of $(\delta_P \leq \delta_Q)$ if $Q \setminus P$ is dense in $\mathbb{R}$. By Theorem 7, $(\delta_P \leq \delta_Q)$ contains no continuous function, iff $Q \setminus P \neq \emptyset$. Also for the special case that if $Q \setminus P$ consists only of a single point we can give a characterization.

Definition 14 *(LLPO)*

$LLPO$ is the set of all functions $S :\subseteq \Sigma^\omega \longrightarrow \Sigma^*$ with

$$S < p, q >= \left\{ \begin{array}{lll} 0 & \text{if } p \in 0^\omega & \text{and } q \neq 0^\omega \\ 1 & \text{if } p \neq 0^\omega & \text{and } q \in 0^\omega \\ \in \{0,1\} & \text{if } p = 0^\omega & \text{and } q = 0^\omega \end{array} \right\} \text{ for all } p, q \in \Sigma^\omega$$

No function $S \in LLPO$ is continuous. The set $LLPO$ corresponds to Brower's "lesser limited principle of omniscience" (Bridges and Richman, [1]).

Theorem 15

Let P, Q be dense in R with $Q \setminus P \neq \emptyset$. Then:

(1) $LLPO \leq (\delta_P \longrightarrow \delta_Q)$,

(2) $LLPO) \equiv (\delta_P \longrightarrow \delta_Q)$ if $Q \setminus P$ has only one point.

Proof:

(1) Let $\nu_P : \mathbb{N} \longrightarrow P$ and $\nu_Q : \mathbb{N} \longrightarrow Q$ be arbitrary bijective numberings. There is some k with $\nu_Q(k) \notin P$. Since P is dense, there is a sequence $[\nu_P(a_i); \nu_P(b_i)]$ of nested intervals converging to $\nu_Q(k)$. Define a function $f : \Sigma^\omega \longrightarrow \Sigma^\omega$ by

$$f < p, q > (j) := \begin{cases} 0 & \text{if } c_m = 0 & \text{and } \nu_P(j) < \nu_P(a_m) \\ 1 & \text{if } c_m = 0 & \text{and } \nu_P(j) \geq \nu_P(a_m) \\ 0 & \text{if } c_m = 1 & \text{and } \nu_P(j) < \nu_P(b_m) \\ 1 & \text{if } c_m = 1 & \text{and } \nu_P(j) \geq \nu_P(b_m) \\ 0 & \text{if } c_m = 2 & \text{and } \nu_P(j) < \nu_P(a_m) \\ 1 & \text{if } c_m = 2 & \text{and } \nu_P(j) \geq \nu_P(a_m) \end{cases}$$

where

$$m := \mu i[p(i) \neq 0 \text{ or } q(i) \neq 0 \text{ or } \nu_P(j) \notin [\nu_P(a_i); \nu_P(b_i)]]$$

and

$$c_m := \begin{cases} 0 & \text{if } p(m) \neq 0 \\ 1 & \text{if } p(m) = 0 \text{ and } q(m) \neq 0 \\ 2 & \text{otherwise.} \end{cases}$$

Then f is continuous.

Since the sequence of closed intervals $[\nu_P(Q_i); \nu_P(b_i)]$ converges to

$\nu_Q(k) \notin P$, for some i $\nu_P(j) \notin [\nu_P(a_i); \nu_P(b_i)]$. Therefore m and c_m exist in any case, i.e. $f < p, q >$ exists for all $p, q \in \Sigma^\omega$.For $p, q \in \Sigma^\omega$ define

$$y(p,q) := \begin{cases} \nu_Q(k) & \text{if } p = q = 0^\omega \text{ else} \\ \nu_P(a_n) & \text{if } p(n) \neq 0 \\ \nu_P(b_n) & \text{if } p(n) = 0 \text{ and } q(n) \neq 0 \end{cases}$$

where $n = \mu i[p(i) \neq 0 \text{ or } q(i) \neq 0]$. We show $\delta_P f < p, q >= y(p,q)$ for all $p, q \in \Sigma^\omega$.

Let $p, q \in \Sigma^\omega$ and $j \in \mathbb{N}$. If $c_m = 0$ then $y(p,q) = \nu_P(a_m)$, if $c_m = 1$ then $y(p,q) = \nu_P(b_m)$, and if $c_m = 2$ then $\nu_P(a_m) < y(p,q) < \nu_P(b_m)$. In any of these cases one shows easily: $\nu_P(i) < y(p,q) \Longrightarrow f < p, q > (i) = 0$ and $\nu_P(i) > y(p,q) \Longrightarrow f < p, q > (i) = 1$. By Definition 6 we have $y(p,q) = \delta_P f < p, q >$. Define $h : \Sigma^\omega \times \Sigma^* \longrightarrow \Sigma^*$ by $h(r,s) := (0$ if $s = 0$, 1 otherwise). h is continuous. Now consider $T \in (\delta_P \longrightarrow \delta_Q)$ and define S by $S(r) := h(r, Tf(r))$. Then $S < p, q >\in \{0, 1\}$ for all $p, q \in \Sigma^\omega$. If $p = 0^\omega$ and $q \neq 0^\omega$ then $\delta_P f < p, q >= y(p,q) = \nu_P(b_n) > \nu_Q(k)$ for some n, hence $\delta_Q T f < p, q >> \nu_Q(k)$ hence $Tf < p, q > (k) = 0$, hence $S < p, q > (k) = 0$. If $p \neq 0^\omega$ and $q = 0^\omega$ we obtain accordingly $S < p, q >= 1$.
This shows that $S \in LLPO$.

(2) Assume $Q \setminus P = \{\nu_Q(k)\}$. We have to show $(\delta_P \longrightarrow \delta_Q) \leq LLPO$. Define $f : \Sigma^\omega \longrightarrow \Sigma^\omega$ by $f(r) :=< p, q >$ where

$$p(n) := \begin{cases} 0 & \text{if } r(n) = 0 \text{ or } \nu_Q(k) < \nu_P(n) \\ 1 & \text{otherwise,} \end{cases}$$

$$q(n) := \begin{cases} 0 & \text{if } r(n) \neq 0 \text{ or } \nu_Q(k) > \nu_P(n) \\ 1 & \text{otherwise.} \end{cases}$$

Then f is continuous.

There is some function $g : \mathbb{N} \longrightarrow \mathbb{N}$ with $\nu_Q(i) = \nu_P g(i)$ for all $i \neq k$. Define $h : \Sigma^\omega \times \Sigma^* \longrightarrow \Sigma^\omega$ by $h(r,w)(k) := w$ and $h(r,w)(i) := rg(i)$ if $i \neq k$. Let S be some function from $LLPO$. Define T by $T(r) := h(r, Sf(r))$. Assume $x = \delta_P(r)$. Then for $i \neq k$, $\nu_Q(i) < x \Longrightarrow \nu_P g(i) < x \Longrightarrow rg(i) = 0 \Longrightarrow T(r)(i) = 0$ and $\nu_Q(i) > x \Longrightarrow T(r)(i) \neq 0$ (accordingly). At Position k we have:

$\nu_Q(k) < x \Longrightarrow p = 0^\omega$ and $q \neq 0^\omega \Longrightarrow Sf(r) = S < p, q >= 0 \Longrightarrow T(r)(k) = 0$,
$\nu_Q(k) > x \Longrightarrow q = 0^\omega$ and $p \neq 0^\omega \Longrightarrow Sf(r) \neq 0 \Longrightarrow T(r)(k) \neq 0$
$\nu_Q(k) = x \Longrightarrow p = q = 0^\omega$ and $Sf(r) \in \{0,1\}$ and $T(r)(k) \in \{0,1\}$.
Therefore we have $T \in (\delta_P \longrightarrow \delta_Q)$ and since $S \in LLPO$ was arbitrary we have $(\delta_P \longrightarrow \delta_Q) \leq LLPO$.
Q.E.D.

A representation $\delta :\subseteq \Sigma^\omega \longrightarrow M$ is admissible, iff $\delta' \leq \delta$ for all (τ_c, τ_δ)–continuous representations $\delta :\subseteq \Sigma^\omega \longrightarrow M$, where τ_c is the Cantor topology on Σ^ω and $\tau_\delta = \{X \subseteq M \mid \delta^{-1}X$ open in $dom\,(\delta)\}$ is the final topology of δ (Kreitz, Weihrauch [4, 10]).The admissible representations can be called "constructively effective". The m–adic representations are not admissible (Kreitz, Weihrauch [10, 9]). We prove this for the broader class of weak separation representations.

Theorem 16

(1) The final topology of any weak separation representation is the real line.

(2) No weak separation representation is admissible.

Proof:

Let $Q \subseteq \mathbb{R}$ be dense and let δ be the derived weak separation representation. We show that the final topology of δ is the real line topology $\tau_{\mathbb{R}}$. We may assume $r \in Dom\,(\delta) \Longrightarrow range\,(r) \subseteq \{0,1\}$.
$\tau_{\mathbb{R}} \subseteq \tau_\delta$: The set $\{(a;b) \in \mathbb{R} \mid a, b \in \mathbb{Q}$ and $a < b\}$ is a basis of $\tau_{\mathbb{R}}$. If suffices to show that $\delta^{-1}(a;b)$ is open in $dom(\delta)$ for $a, b \in Q$, $a < b$. Let $m, n \in \mathbb{N}$ with $\nu(m) < \nu(n)$. Let

$$D := \{q \in \Sigma^\omega \mid \exists m', n' \ : \nu(m) < \nu(m') < \nu(n') < \nu(n) \text{ and } q(m') = 0 \text{ and q(n') = 1}\}$$

Then D is open and $p \in \delta^{-1}(\nu(m), \nu(n)) \Longleftrightarrow p \in D \cap dom(\delta)$.

$\tau_\delta \subseteq \tau_{\mathbb{R}}$: Let $X \subseteq \mathbb{R}$ such that $\delta^{-1}X$ is open in $dom(\delta)$, i.e. $X \in \tau_\delta$. We have to show $X \subseteq \tau_{\mathbb{R}}$. Let $\delta(p) = x \in X$. Then $x \in \delta(w\Sigma^\omega) \subseteq X$ for some prefix w of p.

Let $w = p(0) \dots p(n-1)$.
Case $x \neq \nu(i)$ for $i < n$:
Let $a := max\,\{\nu(i) \mid i < n, p(i) = 0\}$, $b := min\,\{\nu(i) \mid i < n, p(i) \neq 0\}$. Then $x \in (a;b) = \delta(w\Sigma^\omega) \subseteq X$. This shows that x has an open neighbourhood in X.

Case $x = \nu(k)$ for some $k < n$:
Case $p(k) = 0$: Let $b := min\,\{\nu(i) \mid i < n, p(i) = 1\}$, then $[x;b] \subseteq \delta(w\Sigma^\omega) \subseteq X$. Let $p'(k) := 1$ and $p'(i) := p(i)$ for $i \neq k$. Then $\delta(p') = x$ hence $x \in \delta(w'\Sigma^\omega) \subseteq X$ for some prefix w' of p'. As above we show $[a;x] \subseteq \delta(w'\Sigma^\omega)$ for some $a \in \mathbb{R}$. Therefore $x \in (a;b) \subseteq X$, i.e. x has an open neighbourhood in X.

Case $p(k) = 1$: (interchange p with p' in the above case).
Therefore the δ-open set X is open in the real line.

(2) The standard representation δ_{st} of $\mathbb{R}$ is $(\tau_c, \tau_{\mathbb{R}})$-continuous, hence (τ_c, τ_δ)-continuous. By Theorem 7, $\delta_{st} \not\leq \delta$, hence δ is not admissible. Q.E.D.

5. Cut Representations, Iterated Fraction Representations

In Section 3 we have introduced the representation $\delta_<$. $\delta_<(p) = x$ iff p enumerates the set of all i such that $\nu_Q(i) < x$ ("p enumerates the left cut of x").

We shall now use characteristic functions of left cuts as names. For this Section 5 let $Q \subseteq \mathrm{IR}$ be some fixed subset dense in IR and let ν be some numbering of Q. As for weak separation representations the degree of a representation will depend in Q but not on ν.

Definition 17

Define representations $\gamma_<, \gamma_>$ and γ_0 of IR by:

$$\begin{array}{lll} \gamma_<(p) = x & :\Longleftrightarrow p^{-1}\{1\} = \{i \mid \nu(i) < x\} \\ \gamma_>(p) = x & :\Longleftrightarrow p^{-1}\{1\} = \{i \mid \nu(i) > x\} \\ \gamma_0(p) = x & :\Longleftrightarrow \forall i : p_i(i) = \begin{cases} 0 & \text{if } \nu(i) < x \\ 1 & \text{if } \nu(i) = x \\ 2 & \text{if } \nu(i) > x \end{cases} \end{array}$$

If $Q = \mathbb{Q}$ and ν is the standard numbering of $\mathbb{Q}$ then $\gamma_<$ is the left cut representation, $\gamma_>$ is the right cut representation, and γ_0 is (computationally equivalent to) the iterated fraction representation of IR (Deil [2]).

The following lemma can be proved easily (cf. Deil [2]).

Lemma 18

(1) $deg(\gamma_0)$ is the greatest lower bound of $deg(\gamma_<)$ and $deg(\gamma_>)$.

(2) $\gamma_< \not\leq \gamma_0$, $\gamma_> \not\leq \gamma_0$, $\gamma_< \not\leq \gamma_>$, $\gamma_> \not\leq \gamma_<$.

(3) If δ_Q is a weak separation representation then

$$\gamma_< \leq \delta_Q \text{ and } \gamma_> \leq \delta_Q.$$

As in the former cases we introduce a new problem for characterizing the degree of translators for the cut representations.

Definition 19

Let $EC\,1$ be the restriction of EC to $\{p \mid card\,(\mathbb{N} \setminus En(p)) \leq 1\}$, i.e.

$$Cf\,EC\,1(p) = \begin{cases} En(p) & \text{if } card\,(\mathbb{N} \setminus En(p)) \leq 1 \\ div & \text{otherwise.} \end{cases}$$

Lemma 20

$$(\delta_{st} \longrightarrow \gamma_0) \leq \{EC\,1\}$$

Proof:

By Lemma 2 we may use ν for determining δ_{st}:

$\delta_{st} < p, q >= x \Longleftrightarrow En(p) = \{i \mid \nu(i) < x\}$ and $En(q) = \{i \mid \nu(i) > x\}$.

For $< p, q >\in dom\,(\delta_{st})$, $En < p, q >= \{i \mid \nu(i) \neq x\}$. Define $h :\subseteq \Sigma^\omega \times \Sigma^* \longrightarrow \Sigma^*$ by

$$h(< p, q >, 0^i) := \begin{cases} 0 & \text{if } 01^{i+1}0 \text{ is a subword of } p[m] \\ 2 & \text{if } 01^{i+1}0 \text{ is a subword of } q[m] \\ & \text{and not of } p[m] \end{cases}$$

if $m := \mu n[01^{i+1}1$ is a subword of $p[m]$ or of $q[m]]$ exists. For all other arguments let $h(r, w) := div$. Then h is continuous. Define $f :\subseteq \Sigma^\omega \times \Sigma^\omega \longrightarrow \Sigma^\omega$ by

$$f(< p, q >, r)(i) := \begin{cases} h(< p, q >, 0^i) & \text{if } r(i) \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then f is continuous and T with $T < p, q >:= f(< p, q >,$ $EC\,1 < p, q >)$ is a translator from δ_{st} to γ_0. (For $\delta_{st} < p, q >= x$,

$\nu(i) < x \Longrightarrow \ldots \Longrightarrow T < p, q > (i) = 0$, $\nu(i) > x \Longrightarrow T < p, q > (i) = 2$, $\nu(i) = x \Longrightarrow T < p, q > (i) = 1)$.
Q.E.D.

Lemma 21

$$\{EC\,1\} \leq (\gamma_{>} \longrightarrow \gamma_{<})$$

Proof

First we assume that $\nu : \mathbb{N} \longrightarrow Q$ is bijective. For $p \in \Sigma^{\omega}$ and $n \in \mathbb{N}$ define

$$Seg\,(p,n) := max\{k \mid \forall j < k : 01^{j+1}0 \text{ is a subword of } p[n]\}.$$

The sequence $(Seg\,(p,n))$ is unbounded if $En(p) = \mathbb{N}$, otherwise it converges to $min\,(\mathbb{N} \setminus En(p))$. For a given input $p \in \Sigma^{\omega}$ we define $a_n, d_n \in \mathbb{N}$ $(n = 0, 1, \ldots)$ as follows:

$n = 0$

$d_0 := 0$; choose a_0 such that $|\nu(a_0) - \nu(d_0)| > 1$.

$n - 1 \longrightarrow n$

Case $Seg\,(p, n-1) = Seg\,(p,n)$:
Then define $d_n := d_{n-1}$, $a_n = a_{n-1}$
Case $Seg\,(p, n-1) < Seg\,(p,n)$:
Define $d_n := d_{n-1} + 1$;
if $\nu(d_n) > \nu(a_{n-1}) + 2 \cdot 5^{-n}$ then $a_n := a_{n-1}$, otherwise choose a_n such that

$$\nu(a_{n-1}) + 3 \cdot 5^{-n} < \nu(a_n) < \nu(a_{n-1}) + 4 \cdot 5^{-n}.$$

The construction guarantees

$$[\nu(a_n); \nu(a_n) + 5^{-n}] \subseteq [\nu(a_{n-1}); \nu(a_{n-1}) + 5^{n-1}],$$

especially the sequence $(\nu(a_n))_{n \in \mathbb{N}}$ converges to some $x_p \in \mathbb{R}$.

Proposition 1: For all k:

$$(\exists n)k = a_n \text{ or } (\exists n)\nu(k) \notin [\nu(a_n); \nu(a_n) + 5^{-n}]$$

Proof 1: Consider $k \in \mathbb{N}$.
Case $En(p) = \mathbb{N}$: In this case $\{d_0, d_1, \ldots\} = \mathbb{N}$, hence there is some smallest n with $k = d_n$. The construction of a_n guarantees $\nu(k) \notin [\nu(a_n); \nu(a_n) + 5^{-n}]$.
Case $En(p) \neq \mathbb{N}$: then there is some m with $a_n = a_m$ for all $n \geq m$, thus $\nu(a_n) \longrightarrow \nu(a_m)$. Assume $(\forall n)\nu(k) \in [\nu(a_n); \nu(a_n) + 5^{-n}]$. Then $\nu(a_n) \longrightarrow \nu(k)$, hence $\nu(a_m) = \nu(k)$ and $k = a_m$.
q.e.d(1)

Define $f : \Sigma^\omega \longrightarrow \Sigma^\omega$ by

$$f(p)(k) := \begin{cases} 1 & \text{if } \nu(k) > \nu(a_m) \\ 0 & \text{otherwise} \end{cases}$$

where

$$m := \mu n[k = a_n \text{ or } \nu(k) \notin [\nu(a_n); \nu(a_n) + 5^{-n}]].$$

By Proposition 1, $f(p)(k)$ exists for all p, k; f is continuous.
Proposition 2: $x_p := \lim_n \nu(a_n) = \gamma_> f(p)$
Proof (2): For $k \in \mathbb{N}$ let $m = \mu n[\ldots]$ (Def. of f)
Case $k = a_m : \nu(k) \leq x_p \iff \text{true} \iff f(p)(k) = 0$
Case $\nu(k) \notin [\nu(a_m); \nu(a_m) + 5^{-m}]$: Then

$$\nu(k) > x_p \iff \nu(k) > \nu(a_m) \iff f(p)(k) = 1$$

q.e.d.(2)

Define a function $g' :\subseteq \Sigma^\omega \times \Sigma^\omega \times \mathbb{N} \longrightarrow \Sigma$ by

$$g'(p, s, k) := \begin{cases} div & \text{if } h(k) \text{ does not exist, else:} \\ 1 & \text{if } 01^{k+1}0 \text{ is a subword of } p[h(k)] \\ 0 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} h(k) := \mu n \quad & [01^{k+1}0 \text{ subword of } p[n] \\ & \text{or } (Seg(p, n) = k \text{ and } f(p)(a_n) \neq 1 \text{ and } s(a_n) \neq 1] \end{aligned}$$

and define $g :\subseteq \Sigma^\omega \times \Sigma^\omega \longrightarrow \Sigma$ by

$$\begin{aligned} & dom\,(g) = \{(p, s) \mid \forall k : g'(p, s, k) \text{ exists }\} \\ & g(p, s)(k) := g'(p, s, k). \end{aligned}$$

Then g is continuous.
Proposition 3: For any $T \in (\gamma_> \longrightarrow \gamma_<)$ we have $EC\,1(p) = g(p, Tf(p))$ for all $p \in dom\,(EC\,1)$.
Proof 3: For any $p \in \Sigma^\omega$ and $k \in \mathbb{N}$ we have

$$\begin{aligned} & k = Min(\mathbb{N} \setminus En(p)) \\ \Longleftrightarrow\ & (\exists n)[Seg(p,n) = k \text{ and } x_p = a_n] \\ \Longleftrightarrow\ & (\exists n)[Seg(p,n) = k \text{ and } f(p)(a_n) \neq 1 \text{ and } Tf(p)(a_n) \neq 1]. \end{aligned}$$

For each $p \in dom(EC\,1)$ and each $k \in \mathbb{N}$ we have either $(\exists n)01^{k+1}0$ is a subword of $p[n]$ or $k = Min(\mathbb{N} \setminus En(p))$. Therefore, if $p \in dom(EC\,1)$ and $s = Tf(p)$, then for each k one and only one of the alternatives in the definition of $h(k)$ holds. We obtain for $p \in dom(EC\,1)$: $g(p, Tf(p))(k) = 1 \Longleftrightarrow k \in En(p)$.
q.e.d.(3)

$EC\,1 \leq (\gamma_> \longrightarrow \gamma_<)$ follows from Proposition 3. Since the degrees of $\gamma_>$ and $\gamma_<$ do not depend on the particular numbering ν of Q, the assumption that ν is bijective has been made w.l.g.
Q.E.D.

Theorem 22

The following problems are equivalent (δ_Q is any weak separation representation based on Q).

$$\begin{aligned} & \{EC\,1\}, (\delta_{st} \longrightarrow \gamma_0), (\gamma_< \longrightarrow \gamma_>), (\gamma_> \longrightarrow \gamma_<), (\delta_Q \longrightarrow \gamma_>), \\ & (\delta_Q \longrightarrow \gamma_<), (\gamma_< \longrightarrow \gamma_0), (\gamma_> \longrightarrow \gamma_0), (\delta_{st} \longrightarrow \gamma_>), (\delta_{st} \longrightarrow \gamma_<), (\delta_Q \longrightarrow \gamma_0) \end{aligned}$$

Proof:

By Lemma 21, $\{EC\,1\} \leq (\gamma_> \longrightarrow \gamma_<)$. Correspondingly $\{EC\,1\} \leq (\gamma_< \longrightarrow \gamma_>)$ is proved. By Lemma 20. $(\delta_{st} \longrightarrow \gamma_0), \leq \{EC\,1\}$. In general, if $\delta_1 \leq \delta_3$ and $\delta_4 \leq \delta_2$ then $(\delta_1 \longrightarrow \delta_2) \leq (\delta_3 \longrightarrow \delta_4)$. Using this property and the known reducibilities the above equivalences can be proved easily.
Q.E.D.

6. Final Remarks

Most of the translation problems considered in this paper are members of three degrees, the degree of

- $(En \longrightarrow Cf) = \{EC\}$,
- $SEP\,1$, and
- $\{EC\,1\}$.

Easy arguments show

$$SEP\,1 \leq \{EC\,1\} \leq \{EC\}.$$

M. Schröder (private communication) has proved $EC\,1 < EC$, and presumably $SEP\,1 < \{EC\,1\}$ can be shown.

For weak separation representations we have studied translations $(\delta_P \longrightarrow \delta_Q)$ only for the case $Q \setminus P$ dense and the case card $(Q \setminus P) = 1$. The cases $Q \setminus P = k \geq 1$, and $Q \setminus P$ is infinite but non-dense are unsettled. The change of the dense set Q for cut-representations has not been investigated at all.

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