The Degrees of Discontinuity of some Translators between Representations of the Real Numbers

Klaus Weihrauch*

TR-92-050

July 1992

Abstract

Representations like decimal representation are used for defining computability on the set of real numbers. Translatability between different representations has been studied in the past by several authors. Most of the not computably solvable translation problems are not even continuously solvable. In this paper the degrees of discontinuity of translations between a number of common representations are compared and characterized. Mainly three degrees are considered: the first one with translations between the standard representation and the weak cut representations, the second one contains among others the translations between $m$-adic and $n$-adic representations, and the third one contains translations concerning proper cut representations and the iterated fraction representation.

* Fern Universität, D-5800 Hagen, Germany
1. Introduction

Almost every approach to introduce effectivity (constructivity, computability, computational complexity) on the real numbers is (at least indirectly) based on some representation $\delta : \subseteq \Sigma^\omega \rightarrow \mathbb{I}R$, where $\Sigma$ is a finite alphabet and $\Sigma^\omega$ is the set of all infinite sequences of elements of $\Sigma$. If $\delta(p) = x$ then the sequence $p = p(0)p(1) \ldots$ is considered as a name of the real number $x$. A well-known example is the decimal representation $\delta_{10}$, e.g. $\delta_{10}(3,14159 \ldots) = \pi \in \mathbb{I}R$. Representations can be compared by reducibility ("translatability"): $\delta$ is computationally reducible to $\delta'$, $\delta \leq_c \delta'$, if there is some computable function $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$, with $\delta(p) = \delta'f(p)$ for all $p \in \text{dom}(\delta)$. The representations $\delta$ and $\delta'$ are called computationally equivalent, $\delta \equiv_c \delta'$, if $\delta \leq_c \delta'$ and $\delta' \leq \delta$.

The representations on which all the approaches to effectivity in analysis are based are elements of a single equivalence class. It is remarkable that the decimal representation is not in this class. Computational reducibility for a variety of representations if $\mathbb{I}R$ has been studied by T. Deil [2].

The Type 2 Theory of Effectivity (TTE) (see e.g. Kreitz and Weihrauch [3], Weihrauch [10] [12]) gave new insights. Computable functions $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ are continuous (w.r.t. the Cantor topology on $\Sigma^\omega$). This continuity can be interpreted as a very basic kind of constructivity: (a finite portion of the output depends only on a finite portion of the input).

For representations define: $\delta$ is (continuously) reducible to $\delta'$, $\delta \leq \delta'$, iff there is some continuous $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ with $\delta(p) = \delta'f(p)$ for all $p \in \text{dom}(\delta)$. It turns out that for all representations $\delta, \delta'$ considered by Deil [2] either $\delta \leq_c \delta'$ or $\delta \not\leq \delta'$ (while Deil has only proved $\delta \not\leq_c \delta'$). This means that in these cases non-translatability is independent of Church's Theses and depends on a more elementary effectivity concept. Similarly, almost all "effectively unsolvable" problems in Analysis turn out to be not continuously solvable in their TTE-formulation.
In ordinary recursion theory, "computable" reducibilities and corresponding degrees of unsolvability are introduced for comparing and classifying the kinds of non-effectivity of sets and functions. Correspondingly, in TTE continuous reducibilities can be used for comparing and classifying kinds of non-continuity (i.e. non-constructivity) of problems in Analysis (v. Stein [7], Mylatz [5]). While for subsets of $\Sigma^0_\omega$ a nice reducibility theory is available (Wadge [8], see also Weihrauch [11] and Staiger/Weihrauch [6]), for functions $f : \subseteq \Sigma^\omega\rightarrow \Sigma^\omega$ we have a natural reducibility but almost no knowledge about its structure. Before developing a theory also for this case, more non trivial examples should be studied in detail.

In this paper we introduce a concept of continuous reducibility for "problems" and characterize the degrees of discontinuity for the translation problems between several representations of the real numbers.

The non-continuous translation problems between the standard representation, the left-cut and right-cut r.e. representations and the naive Cauchy representation belong to a single degree. This degree contains the translation from the enumeration representation to the characteristic function representation of the natural numbers.

The $m$-adic representations are special cases of the "weak separation" representations. It is shown that the non-continuous translation problems between most pairs of weak separation representations (including the $m$-adic representations) also belong to a single degree of discontinuity which is characterized by a special separation problem of open sets.

Finally, the non-continuous translation problems concerning right-cut, left-cut and continued fraction representations belong to a single degree which can also be characterized by a simple abstract problem.
2. Prerequisites and Basic Definitions

Let $\Sigma$ be some sufficiently large finite alphabet with $\{0, 1\} \subseteq \Sigma$. By $\Sigma^*$ we denote the finite sequences (words), by $\Sigma^\omega$ the infinite sequences over $\Sigma$. Concatenation of words $x, y \in \Sigma^*$ and sequences $p \in \Sigma^\omega$ is denoted by $xy \in \Sigma^*$ and $xp \in \Sigma^\omega$, resp. For $X, Y \subseteq \Sigma^*$, and $T \subseteq \Sigma^\omega$, define $XT := \{xt \mid x \in X, t \in T\}$ (correspondingly for $xY, XY, \ldots$).

On $\Sigma^*$ we consider the discrete topology (every $X \subseteq \Sigma^*$ is open). On $\Sigma^\omega$ we consider the “Cantor topology” $\tau_C$ defined by the following basis of open sets:

$$\{w\Sigma^\omega \mid w \in \Sigma^*\}.$$  

The standard pairing $\langle, \rangle: \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega$ is defined by $\langle p, q \rangle := (p(0), q(0), p(1), q(1), \ldots)$. Tuplings with more than two arguments are defined accordingly. Tuplings and their inverses are continuous. The standard (Cantor) pairing $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is also denoted by $\langle, \rangle$ (accordingly for tuplings with more than two arguments).

A representation of a set $M$ is a possibly partial, surjective function

$$\delta : \subseteq \Sigma^\omega \rightarrow M.$$  

The following representations will be used repeatedly (see Weihrauch [10]):

$$En : \Sigma^\omega \rightarrow 2^\omega$$

(enumeration representation)

$$En(p) := \{n \mid 01^{n+1}0 \text{ is a substring of } p\},$$

$$ Cf : \Sigma^\omega \rightarrow 2^\omega$$

(characteristic function representation)

$$Cf(p) := \{n \mid p(n) = 1\},$$

$$t : \subseteq \Sigma^\omega \rightarrow \omega^\omega$$

$$\bar{p} := t(p)$$

$$t^{-1}(n_0 n_1 \ldots) := 1^{n_0}01^{n_1}0 \ldots$$

For $p \in \Sigma^\omega$ let $p[n] := p(0) \ldots p(n-1) \in \Sigma^*$, and for $w \in \Sigma^*$ let $En(w) := \{n \mid 01^{n+1}0 \text{ is a subword of } w\}$.

Continuous reducibility (reducibility for short) for representations is defined by

$$\delta \leq \delta' : \iff \exists f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \text{ continuous: } \forall p \in dom(\delta) : \delta(p) = \delta'(f(p)),$$
\( \delta < \delta' \) means \( \delta \leq \delta' \) and not \( \delta' \leq \delta \). Let

\[
(\delta \rightarrow \delta') = \{ f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \mid \forall p \in \text{dom}(\delta) : \delta(p) = \delta'f(p) \}
\]

be the set of all (not necessarily continuous) translators from \( \delta \) to \( \delta' \).

For functions \( f, g : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) (or \( \Sigma^* \)) define reducibility by

\[
f \leq g : \iff \exists \text{ continuous functions } A, B : \forall p \in \text{dom}(f) : f(p) = A(p, gB(p)).
\]

A set \( X \) of functions \( f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) (or \( \Sigma^* \)) may be considered as the set of solutions of a problem \( P_X \) (e.g., the set \( (\delta \rightarrow \delta') \) is the set of solutions of the problem to translate \( \delta \) to \( \delta' \)). We compare the "difficulty" of such problems by a uniform reducibility. For sets \( X, Y \) of functions \( : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega \) (or \( \Sigma^* \)) define:

\[
X \leq Y : \iff \text{ there are continuous functions } A, B :
\forall g \in Y : \exists f \in X : \forall p \in \text{dom}(f) : f(p) = A(p, gB(p))
\]

Thus \( A \) and \( B \) define a uniform method which for every solution \( g \) of Problem \( P_Y \) produces a solution of Problem \( P_X \). Clearly \( f \leq g \iff \{f\} \leq \{g\} \), hence \( f \leq g \) may be considered as a short notation of \( \{f\} \leq \{g\} \). The relation \( \leq \) is transitive. As usually, define \( X \equiv Y : \iff X \leq Y \text{ and } Y \leq X \) (equivalence). The equivalence classes can be called "discontinuity degrees of problems".
3. Representations Below the Standard Representation

Let $Q \subseteq \mathbb{R}$ be some dense subset of $\mathbb{R}$, and let $\nu_Q : \mathbb{N} \rightarrow Q$ be some (total) numbering of $Q$. We define four representations $\subseteq \Sigma^\omega \rightarrow \mathbb{R}$ of the reals numbers.

Definition 1

(1) Naive Cauchy representation $\delta_C$:

$$\delta_C(p) = x \iff \text{the sequence } \nu_Q p(0), \nu_Q p(1), \ldots \text{ converges to } x$$

for all $p \in \Sigma^\omega$ and $x \in \mathbb{R}$.

(2) R.e. left cut representation $\delta_<$:

$$\delta_<(p) = x \iff En(p) = \{ i \mid \nu_Q(i) < x \}$$

for all $p \in \Sigma^\omega$ and $x \in \mathbb{R}$.

(3) R.e. right cut representation $\delta_>$:

$$\delta_>(p) = x \iff En(p) = \{ i \mid \nu_Q(i) > x \}$$

for all $p \in \Sigma^\omega$ and $x \in \mathbb{R}$.

(4) Standard representation $\delta_{st}$:

$$\delta_{st} < p, q > = x \iff \delta_<(p) = \delta_>(q) = x$$

for all $p, q \in \Sigma^\omega$ and $x \in \mathbb{R}$.

For the special case that $\nu_Q$ is a standard numbering of the rational numbers, e.g., $\nu < i, j, k > := (i - j) \setminus (1 + k)$, translatability has been discussed in detail by Deil [2] (computational) and Weihrauch [10] (continuous). For this $\nu_Q$, the representation $\delta_{st}$ is computationally equivalent to those representations of the real numbers on which almost all approaches to effective analysis are based.

The degrees of $\delta_C, \delta_<, \delta_>$ and $\delta_{st}$, do not depend on the particular choice of $Q$ and $\nu_Q$. 
Lemma 2
Let \( R \) be dense in \( \mathbb{R} \) and let \( \nu_R : \mathbb{N} \rightarrow R \) be a numbering of \( R \). Define \( \delta_C', \delta_<', \delta_>', \) and \( \delta_{st}' \) by replacing \( \nu_R \) for \( \nu_Q \) in Definition 1. Then
\[
\delta_C' \equiv \delta_C, \quad \delta_<' \equiv \delta_<, \quad \delta_> \equiv \delta_>, \quad \delta_{st}' \equiv \delta_{st}.
\]

Proof:
\( \delta_C \leq \delta_C' \): There is a function \( h : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) with \( |\nu_Q(i) - \nu_R(h(i,j))| < 2^{-j} \). Define a function \( f : \Sigma^\omega \rightarrow \Sigma^\omega \) by \( \text{dom}(f) := \text{dom}(i) \) and
\[
f(1^{n_0}01^{m_1}0 \ldots) := 1^{h(n_0)}01^{h(m_1)}0 \ldots.
\]
Then \( f \) is continuous and \( \delta_C(p) = \delta_C f(p) \) for all \( p \in \text{dom}(\delta_C) \).

\( \delta_< \leq \delta_<' \): For given \( p \in \Sigma^\omega \) define words \( w_i \in \Sigma^* (i = 0, 1, \ldots) \) by
\[
w_{<i,k,m>} := \begin{cases} 
01^{m+1}0 & \text{if } \nu_R(m) < \nu_Q(k) \text{ and } 01^{k+1}0 \text{ is a subword of } p(0) \ldots p(i) \\
0 & \text{otherwise}
\end{cases}
\]
and set \( f(p) := w_0w_1 \ldots \). Then \( f \) is continuous and
\[
m \in \text{En } f(p) \iff \exists k \in \text{En}(p) : \nu_R(m) < \nu_Q(k),
\]
hence \( \delta_< (p) = \delta_< f(p) \).
The other statements are proved correspondingly.
Q.E.D.

The following translatability results are well known [2, 10]:

Theorem 3
(1) \( \delta_{st} < \delta_< < \delta_C, \quad \delta_{st} < \delta_> < \delta_C \),
(2) \( \delta_< \not\leq \delta_> \) and \( \delta_\not\geq \delta_< \).
(3) $\delta_{st}$ is (up to equivalence) the least upper bound of $\delta_<$ and $\delta_>$.

The proofs are easy. For illustration two examples are included.

**Proof**

$\delta_\leq \leq \delta_C$: For $p \in \Sigma^\omega$ define a sequence $w_0, w_1, \ldots$ of words as follows:

\[
\begin{align*}
  w_0 &:= \varepsilon \\
  w_n &:= \begin{cases} 
    0 & \text{if } E_n(p[n]) = \emptyset \\
    01^{k+1}0 & \text{if } k \in E_n(p[n]) \text{ and } \\
    & \nu_Q(m) \leq \nu_Q(k) \text{ for all } m \in E_n(p[n]).
  \end{cases}
\end{align*}
\]

Define $f(p) := w_0 w_1 \ldots$. Then $f : \Sigma^\omega \to \Sigma^\omega$ is continuous and $\delta_\leq(p) = \delta_C f(n)$ for all $p \in \text{dom}(\delta_\leq)$.

$\delta_\leq \not\leq \delta_>$: Assume, $\delta_\leq \leq \delta_>$. Then there is some continuous $f : \subseteq \Sigma^\omega \to \Sigma^\omega$ with $\delta_\leq(p) = \delta_>(p)$ for all $p \in \text{dom}(\delta_\leq)$. Let $p = 01^m 01^n 0 \ldots$ with $\delta_\leq(p) = x$. Then $\delta_>(p) = x$. Hence for some $k, n 01^{k+10}$ is a suffix of $f(p)[n]$. Consequently, $x < \nu_Q(k)$. By continuity of $f$, there is a prefix $v$ of $p$ such that $f(v \Sigma^\omega) \subseteq f(p)[n] \Sigma^\omega$. There is some $r \in \Sigma^\omega$ such that $\delta_<(vr) > \nu_Q(k)$, but $f(vr) \notin f(p)[n] \Sigma^\omega$, hence $\nu_Q(k) > \delta_>(vr) = x$ (contradiction).

Q.E.D.

By the next theorem the essential parts of which are from v. Stein [7] all the not continuously solvable translation problems between the above representations are in the same degree of discontinuity, namely in the degree of $EC : \Sigma^\omega \to \Sigma^\omega$, the translator from the enumeration representation $E_n$ to the characteristic function representation $C_f$ of $2^\omega$ (i.e. $\{EC\} = (E_n \to C_f)$). Notice that $EC$ is a ”very non continuous” function.
Theorem 4

The following sets of translations are equivalent:

\( (En \rightarrow Cf), (\delta_C \rightarrow \delta_\prec), (\delta_C \rightarrow \delta_\succ), (\delta_\prec \rightarrow \delta_{st}), (\delta_\succ \rightarrow \delta_{st}), (\delta_\succ \rightarrow \delta_\prec), (\delta_\prec \rightarrow \delta_{st}), (\delta_{st} \rightarrow \delta_{st}). \)

Proof

Proposition 1: \( \{ EC \} \leq (\delta_\prec \rightarrow \delta_\succ), \{ EC \} \leq (\delta_\succ \rightarrow \delta_\prec). \)

Proof 1: For \( p \in \Sigma^\omega \) define a sequence \( w_0, w_1, \ldots \) of words by

\[
w_{<k,m>} := \begin{cases} 01^{m+1}0 & \text{if } \nu_Q(m) < \sum \{3^{-i} \mid i \in En(p[k])\} \\ 0 & \text{otherwise} \end{cases}
\]

and define \( f(p) := w_0, w_1, \ldots \). Then \( f \) is continuous, and

\[
\delta_\prec f(p) = \sum \{3^{-i} \mid i \in En(p)\}
\]

for all \( p \in \Sigma^\omega \). Define a function \( g : \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega \) as follows.

For arguments \( r, s \in \Sigma^\omega \) define a sequence \( c_k \) by the following recursion.

\( k = 0:\)

Choose some \( m_0 \) with \( 1/2 < \nu_Q(m_0) < 1 \). Define

\[
j_0 := \mu j[m_0 \in En(r[j]) \text{ or } m_0 \in En(s[j])]
\]

\[
c_0 := \begin{cases} div & \text{if } j_0 \text{ does not exist} \\ 1 & \text{if } m_0 \in En(r[j_0]) \\ 0 & \text{otherwise.} \end{cases}
\]

\( k - 1 \rightarrow k:\)

If \( c_{k-1} \) does not exist then \( c_k \) does not exist. Assume \( c_0, \ldots, c_{k-1} \) have been determined. Define

\[
a_k := \sum \{3^{-i} \mid i < k, c(i) = 1\}
\]

and choose some \( m_k \) with

\[
a_k + 3^{-k}/2 < \nu_Q(m_k) < a_k + 3^{-k}.
\]
Define
\[ j_k := \mu j[m_k \in En(r[j]) \text{ or } m_k \in En(s[j])] \]
\[ c_k := \begin{cases} \text{div} & \text{if } j_k \text{ does not exist} \\ 1 & \text{if } m_k \in En(r[j_k]) \\ 0 & \text{otherwise.} \end{cases} \]

Define \( g(r,s) = c_0 c_1 \ldots \) if \( c_i \) exists for all \( i \), \( g(r,s) = \text{div} \) otherwise. Then \( g \) is continuous.

Proposition 2: Assume \( \delta_<\{r\} = \delta_\geq\{s\} = \sum\{3^{-i} \mid i \in En(p)\} \Rightarrow x_p \).

Then for all \( k \) \( g(r,s)(k) \) exists and \( k \in En(p) \iff g(r,s)(k) = 1 \).

Proof 2: By assumption \( En(r) \cap En(s) = \emptyset \). Let \( m_0, m_1, \ldots \) be the numbers of the above construction for \( r \) and \( s \).

\( k = 0 \):
\( 0 \in En(p) \Rightarrow x_p \geq 1 \Rightarrow v_Q(m_0) < x_p \Rightarrow m_0 \in En(r) \setminus En(s) \Rightarrow g(r,s)(0) = 1 \)
\( 0 \notin En(p) \Rightarrow x_p \leq 1/2 \Rightarrow v_Q(m_0) > x_p \Rightarrow m_0 \in En(s) \setminus En(r) \Rightarrow g(r,s)(0) = 0 \)

\( k - 1 \rightarrow k \):
By induction, \( c_0, \ldots, c_{k-1} \) exists. We obtain
\( k \in En(p) \Rightarrow x_p \geq a_k + 3^{-k} \Rightarrow v_Q(m_k) < x_p \Rightarrow m_k \in En(r) \setminus En(s) \Rightarrow g(r,s)(k) = 1 \)
\( k \notin En(p) \Rightarrow g(r,s)(k) = 0 \) (accordingly)

q.e.d. (2)

Define \( h(p,q) := g(f(p),q) \). Then \( h \) is continuous.

Let \( T \in (\delta_< \longrightarrow \delta_>) \). Then for any \( p \in \Sigma^\omega \), \( \delta_\leq f(p) = \delta_\geq T f(p) = \sum\{3^{-i} \mid i \in En(p)\} \), hence by Proposition 2, \( g(f(p),T f(p)) \) exists and \( \forall k : k \in En(p) \iff g(f(p),T f(p))(k) = 1 \iff h(p,T f(p))(k) = 1 \). Therefore, for all \( T \in (\delta_< \longrightarrow \delta_>) \) \( EC(p) = h(p,T f(p)) \), hence \( \{EC\} \leq (\delta_< \longrightarrow \delta_>) \). \( \{EC\} \leq (\delta_\geq \longrightarrow \delta_<) \) can be proved accordingly.

q.e.d. (1)

Proposition 3: \( (\delta_C \longrightarrow \delta_{st}) \leq \{EC\} \).
Proof 3: For $p \in \Sigma^\omega$ define a sequence $w_0, w_1, \ldots$ of words by:

$$w_{<i,n,k>} := \begin{cases} 01^{1+<n,k>} & \text{if } i > k \text{ and } |\nu_Q \overline{p}(k) - \nu_Q \overline{p}(i)| \geq 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

and set $f(p) := w_0 w_1 \ldots$. Then $f$ is continuous and

$$En f(p) = \{< n, k > \mid \exists i > k : |\nu_Q \overline{p}(k) - \nu_Q \overline{p}(i)| \geq 2^{-n}\},$$

therefore,

$$EC f(p) < n, k > \equiv (\forall i > k) |\nu_Q \overline{p}(k) - \nu_Q \overline{p}(i)| < 2^{-n}.$$

For $p, r \in \Sigma^\omega$ define sequences $x_i, y_i (i \in \mathbb{N})$ of words by

$$x_{<m,k,n>} := \begin{cases} 01^{m+1}0 & \text{if } r < n, k > \equiv 1 \text{ and } \nu_Q(m) < \nu_Q \overline{p}(k) - 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

$$y_{<m,k,n>} := \begin{cases} 01^{m+1}0 & \text{if } r < n, k > \equiv 1 \text{ and } \nu_Q(m) > \nu_Q \overline{p}(k) + 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

and define $g_1(p, r) := x_0 x_1 \ldots$ and $g_2(p, r) := y_0 y_1 \ldots$.

Then $g_1$ and $g_2$ be continuous. If $\delta_C(p) = x$ and $r = EC f(p)$ then $En(g_1(p, r)) = \{m \mid \nu_Q(m) < x\}$ and $En(g_2(p, r)) = \{m \mid \nu_Q(m) > x\}$. Define $h$ by $h(p, r) := < (g_1(p, r), g_2(p, r) >$. Then $h$ is continuous and the function $t$ with $t(p) = h(p, EC f(p))$ translates $\delta_C$ into $\delta_{st}$.

Q.E.D. (3)

From Theorem 3 and Proposition 1 und 3 we obtain

$$\{EC\} \leq \left( \begin{array}{c} (\delta_\prec \to \delta_\succ) \leq \left( \begin{array}{c} (\delta_C \to \delta_\succ) \\ (\delta_\prec \to \delta_{st}) \end{array} \right) \\ (\delta_\succ \to \delta_\prec) \leq \left( \begin{array}{c} (\delta_C \to \delta_\prec) \\ (\delta_\succ \to \delta_{st}) \end{array} \right) \end{array} \right) \leq (\delta_C \to \delta_{st}) \leq \{EC\}.$$
4 The class of m-adic and Related Representations

Reducibility between m-adic representations has been investigated in the past by many authors (see Deil [2]). Here we shall characterize the degree of the non continuously solvable translation problems.

**Definition 5** (m-adic representation)

Let $\Sigma$ be a finite alphabet and $m \in \mathbb{N}$, $m \geq 2$ with $\{0,1,\ldots,m-1,1,-1,\ldots\} \subseteq \Sigma$. The m-adic representation $\delta_m : \Sigma^\omega \to \mathbb{IR}$ of $\mathbb{IR}$ is defined by

$$\delta_m(sa_n \ldots a_0 a_{-1} a_{-2} \ldots) = s \cdot \sum \{a_i m^i \mid i \leq n\}$$

where $s \in \{+1, -1\}$ and $a_i \in \{0, \ldots, m - 1\}$ for $i \leq n$. $\delta_m(p)$ is undefined for arguments $p$ which do not have this form.

First, we embed the m-adic representation into a more general class.

**Definition 6** (weak separation representation)

Let $Q \subseteq \mathbb{IR}$ be dense in $\mathbb{IR}$ and let $\nu : \mathbb{N} \to Q$ be a numbering of $Q$. Define $\delta_\nu : \Sigma^\omega \to \mathbb{IR}$ by

$$\delta_\nu(r) = x :\iff \forall i \begin{cases} r(i) = 0 & \text{if } \nu_Q(i) < x \\ r(i) \neq 0 & \text{if } \nu_Q(i) > x \end{cases}$$

for all $r \in \Sigma^\omega$ and $x \in \mathbb{IR}$.

Notice that $r(i)$ is arbitrary if $\nu(i) = x$.

Reducibility between weak separation representations can be characterized easily.
Theorem 7

Let $Q, R$ be dense in $\mathbb{R}$, let $\nu : \mathbb{N} \rightarrow Q$, $\mu : \mathbb{N} \rightarrow R$, be numberings and let $\delta_\nu$ and $\delta_\mu$ be the corresponding weak separation representations. Let $\delta_{st}$ be the standard representation (Def.1). Then:

1. $\delta_\nu < \delta_{st}$,
2. $R \subseteq Q \iff \delta_\nu \leq \delta_\mu$.

Proof

First we prove (2).

Consider the case $R \subseteq Q$. Then there is some function $h : \mathbb{N} \rightarrow \mathbb{N}$ with $\mu(i) = \nu h(i)$ for all $i$. Define $f : \Sigma^\omega \rightarrow \Sigma^\omega$ by $f(r)(i) := rh(i)$ for all $r \in \Sigma^\omega$ and $i \in \mathbb{N}$. Then $f$ is continuous. If $\delta_\nu(r) = x$ we obtain for all $i$:

$$\mu(i) < x \implies \nu h(i) < x \implies rh(i) = 0 \implies f(r)(i) = 0,$$
$$\mu(i) > x \implies \nu h(i) > x \implies rh(i) \neq 0 \implies f(r)(i) \neq 0,$$

hence $\delta_\nu(r) = x = \delta_\mu f(r)$. This proves $\delta_\nu \leq \delta_\mu$.

If especially $R = Q$ then for any two numberings $\nu$ and $\mu$ of $Q$ we have $\delta_\nu \equiv \delta_\mu$.

Consider now the case $R \not\subseteq Q$. Let us assume first that $\nu : \mathbb{N} \rightarrow Q$ and $\mu : \mathbb{N} \rightarrow R$ are bijective. By assumption there is some $k$ with $x := \mu(k) \in R \setminus Q$.

Since $x \not\in Q$, for all $i$ either $\nu(i) < x$ or $\nu(i) > x$, hence by injectivity of $\nu$ there is only one $p \in \Sigma^\omega$ with $\delta_\nu(p) = x$. If $\delta_\mu(q) = x$, then $q(i)$ is uniquely defined for all $i \neq k$ since $x = \mu(k)$ und $\mu$ is injective. Now let $T : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ be translator from $\delta_Q$ to $\delta_R$, i.e. $T \in (\delta_Q \rightarrow \delta_R)$. Assume that $T$ is continuous. We have $x = \delta_\nu(p) = \delta_\mu T(p)$.

Case $T(p)(k) = 0$:

Let $v \in \Sigma^\omega$ be the prefix of $T(p)$ of length $k$. Then $T(p) \in v0\Sigma^\omega$. By continuity of $T$ there is some prefix $u$ of $p$ such that $T(u\Sigma^\omega) \subseteq v0\Sigma^\omega$. 

13
Let \( n := \text{length}(u) \). Since \( \nu(i) \neq x \) for all \( i \), we have
\[
a := \max \{ \nu(i) \mid i < n, p(i) = 0 \} < x < \min \{ \nu(i) \mid i < n, p(i) \neq 0 \} =: b
\]

There is some \( y \in \mathbb{R} \) with \( a < y < x \). Let \( p' \in \Sigma^\omega \) with \( \delta_\nu(p') = y \). Since \( a < y < x \), obviously \( p' \in u\Sigma^\omega \), hence \( T(p') \in u\Sigma^\omega \) and \( T(p')(k) = 0 \).

On the other hand \( \delta_\mu T(p') = y < x = \mu(k) \), hence \( T(p')(k) = 0 \) by definition of \( \delta_\mu \) (contradiction).

**Case** \( T(p)(k) \neq 0 \):

A contradiction can be derived accordingly. Therefore, there is no continuous translator \( T \), i.e. \( \delta_\nu \not\leq \delta_\mu \).

Now let \( \nu : \mathbb{N} \rightarrow Q \) and \( \mu : \mathbb{N} \rightarrow R \) be arbitrary and \( R \subseteq Q \). There are bijective numberings \( \nu' : \mathbb{N} \rightarrow Q \) and \( \mu' : \mathbb{N} \rightarrow R \). From the first part of our proof we know \( \delta_\nu \equiv \delta_\nu' \) and \( \delta_\mu \equiv \delta_\mu' \), from the second part \( \delta_\nu \not\leq \delta_\mu' \), hence \( \delta_\nu \not\leq \delta_\mu \).

Finally, we prove (1).

For any \( p \in \Sigma^\omega \) define words \( w_0, w_1, \ldots \in \Sigma^* \) by
\[
w_{<i,j>} := \begin{cases} 01^{j+1}0 & \text{if } \nu(j) < \nu(i) \text{ and } p(i) = 0 \\ 0 & \text{otherwise,} \end{cases}
\]

and define \( f_<(p) = w_0w_1 \ldots \). Then \( f_< \) is continuous and \( \delta_\nu(p) = \delta_\nu f_<(p) \) for all \( p \in \text{dom}(\delta_Q) \). Correspondingly there is a continuous translation \( f_> \) from \( \delta_\nu \) to \( \delta_\mu \). Define \( f(p) := f_<(p), f_>(p) > \). Then \( f \in (\delta_\nu \rightarrow \delta_\mu) \). This shows \( \delta_\nu \leq \delta_\mu \). Assume \( \delta_\mu \leq \delta_\nu \). Then for all numberings \( \mu : \mathbb{N} \rightarrow R \), \( R \) dense in \( \mathbb{R} \): \( \delta_\mu \leq \delta_\mu \leq \delta_\nu \) which is false by (2) of this theorem.

Q.E.D.

Since the equivalence class of \( \delta_\nu \) does not depend on the particular numbering \( \nu : \mathbb{N} \rightarrow Q \) of \( Q \), we shall use the notation \( \delta_Q \) for \( \delta_\nu \) considering some fixed numbering implicitly.

There are several interesting representations which are equivalent to weak separation representations (Deil [2]). We prove this only for the \( m \)-adic representations.
Theorem 8

Let $\delta_m$ be the $m$-adic representation and let $\nu < i, j, k > := (i - j)/m^k$ and $Q := \text{range}(\nu)$. Then

$$\delta_Q \equiv \delta_m.$$  

Proof:

$\delta_m \leq \delta_Q$:

For $p = +1a_n \ldots a_0 \cdot a_{-1} \ldots$ define

$$\Gamma_+(p) < i, j, k > := \begin{cases} 0 & \text{if } \nu < i, j, k > \leq a(p, k) \\ 1 & \text{if } \nu < i, j, k > > a(p, k) \end{cases}$$

where $a(p, k) := (a_n \ldots a_0 \cdot a_{-1} \ldots a_{-k})_m := \sum \{a_i m^i \mid n \geq i \geq -k\}$. Then $\Gamma_+$ is continuous on $+1\Sigma^w \cap \text{dom}(\delta_m)$. For all $< i, j, k >$ and all $p \in +1\Sigma^w \cap \text{dom}(\delta_m)$ we have

$$\nu < i, j, k > < \delta_m(p) \implies \nu < i, j, k > \leq a(p, k) \implies \Gamma_+(p) < i, j, k > = 0$$
$$\nu < i, j, k > > \delta_m(p) \implies \nu < i, j, k > > a(p, k) \implies \Gamma_+(p) < i, j, k > \neq 0$$

Therefore $\delta_Q \Gamma_+(p) = \delta_m(p)$. Accordingly there is a continuous function $\Gamma_-$ with $\delta_Q \Gamma_-(p) = \delta_m(p)$ for all $p \in -1\Sigma^w \cap \text{dom}(\delta_m)$. Therefore $\Gamma$ with $\Gamma(p) = \Gamma_+(p)$ if $p(0) = +1$, $\Gamma_-(p)$ otherwise) is a continuous translator from $\delta_m$ to $\delta_Q$.

$\delta_Q \leq \delta_m$:

If $\delta_Q(p) = x$ and $p(0) = 0$ then $\nu(0) \leq x$, hence $0 = \nu < 0, 0, 0 > = \nu(0) \leq x$. First, for a given $p \in \text{dom}(\delta_Q)$ with $p(0) = 0$ we determine $n$ and $a_n, a_{n-1}, \ldots$ such that $\delta_Q(p) = \delta_m(+1a_n a_{n-1} \ldots a_0 \cdot a_{-1} \ldots)$.

Step 0: Let $i_0$ be the number with $p < i_0, 0, 0 > = 0$ and $p < i_0 + 1, 0, 0 > \neq 0$. Then

$$i_0 = \nu < i_0, 0, 0 > \leq \delta_Q(p) \leq \nu < i_0 + 1, 0, 0 > = i_0 + 1.$$  

Determine $n$ and $a_n, \ldots, a_0 \in \{0, \ldots, m-1\}$ such that $i_0 = (a_n \ldots a_0)_m$.  

15
Step \( k+1 \): Assume that \( a_0, \ldots, a_{-k} \) and \( b_k \) and \( c_k \) have been determined with
\[
\nu(b_k) = (a_m \ldots a_0 \cdot a_1 \ldots a_{-k})_m \leq \delta_Q(p) \leq (a_m \ldots a_0 \cdot a_{-1} \ldots a_{-k})_m + m^{-k} = \nu(c_k)
\]
and \( p(b_k) = 0 \) and \( p(c_k) \neq \emptyset \). Define \( d_0 := b_k \), \( d_m := c_k \), and for \( j = 1, \ldots, m - 1 \) choose \( d_j \) such that \( \nu(d_j) = \nu(d_0) + j \cdot m^{-(k+1)} \). Let \( a_{-(k+1)} \) be the greatest \( j \) with \( p(d_j) = 0 \) and let \( b_{k+1} := d_j \) and \( c_{k+1} := d_{j+1} \) for this \( j \). Then
\[
\nu(b_{k+1}) = (a_m \ldots a_0 \cdot a_{-1} \ldots a_{-(k+1)})_m \leq \delta_Q(p) \leq (a_m \ldots a_0 \cdot a_{-1} \ldots a_{-(k+1)})_m + m^{-(k+1)} = \nu(c_{k+1})
\]
and \( p(b_{k+1}) = 0 \) and \( p(c_{k+1}) = 1 \).

By this procedure a continuous function \( f_+ : \Sigma^\omega \rightarrow \Sigma^\omega \) is defined such that for all \( p \in 0\Sigma^\omega \cap \text{dom} (\delta_Q) : \delta_Q(p) = \delta_m f_+(p) \). Accordingly, there is a continuous function \( f_- \) such that \( \delta_Q(p) = \delta_m f_-(p) \) if \( p \in \text{dom}(\delta_Q) \setminus 0\Sigma^\omega \). A combination of \( f_+ \) and \( f_- \) gives a continuous translator from \( \delta_Q \) to \( \delta_m \).

Q.E.D.

As a corollary of Theorem 7 and 8 we obtain the well known translatability result for \( m \)-adic representations [2, 10].

**Corollary 9**

For all \( m, n \geq 2 \):

1. \( \delta_m < \delta_{st} \),
2. \( n \) divides a power of \( m \iff \delta_m \leq \delta_n \).

**Proof:**

1. \( \delta_m \equiv \delta_Q < \delta_{st} \)
2. Let \( Q_m := \{(i - j)/m^k \mid i, j, k \in \mathbb{N} \} \), \( Q_n := \{(i - j)/n^k \mid i, j, k \in \mathbb{N} \} \).
Then $\delta_m \leq \delta_n \iff \delta_{Q_m} \leq \delta_{Q_n} \iff Q_n \subseteq Q_m \iff n$ divides a power of $m$.

Q.E.D.

Our next aim is to characterize degrees of translators between weak separation representations. For this purpose we introduce a problem which is not formulated in terms of the reals numbers.

Definition 10 (1-separation problem)

Let SEP1 be the set of all functions $f : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega$ such that for all $p, q \in \Sigma^\omega$ with $\text{En}(p) \cap \text{En}(q) = \emptyset$ and $\text{card}(\mathbb{N} \setminus (\text{En}(p) \cup \text{En}(q))) \leq 1$ $f < p, q >$ exists and

$$\forall i : f < p, q > (i) = \begin{cases} 0 & \text{if } i \in \text{En}(p) \\ 1 & \text{if } i \in \text{En}(q). \end{cases}$$

Thus the set $X := (f < p, q >)^{-1}\{0\}$ separates $\text{En}(p)$ and $\text{En}(q)$, since $\text{En}(p) \subseteq X \subseteq \mathbb{N} \setminus \text{En}(q)$. We shall call SEP1 the 1-separation problem. Notice the formal similarity between the 1-separation problem and the problem of finding $\delta_Q$-names.

Lemma 11

Let $Q$ be dense in $\mathbb{R}$, let $\nu : \mathbb{N} \rightarrow Q$ be a bijective numbering and let $\delta_{st}$ be the derived standard representation (Def.1) and $\delta_Q$ the derived weak separation representation. Then

$$(\delta_{st} \rightarrow \delta_Q) \leq \text{SEP1}$$

Proof

Let $S \in \text{SEP1}$ be a separator. Assume $< p, q > \in \text{dom}(\delta_{st})$, $x := \delta_{st} < p, q >$. Then $\text{En}(p) \cap \text{En}(q) = \emptyset$, $\text{En}(p) \cup \text{En}(q) = \mathbb{N}$ if
\[ \delta_{st} < p, q > \in Q, \text{En}(p) \cup \text{En}(q) = \mathbb{N} \setminus \{k\} \text{ for the } k \text{ with } \nu_Q(k) = \delta_{st} < p, q > \text{ otherwise. Remember that } \nu \text{ is injective. Therefore, } \langle p, q \rangle \in \text{dom}(S) \text{ and for all } i:\]

\[
\begin{align*}
\nu_Q(i) < x & \implies i \in \text{En}(p) \implies S < p, q > (i) = 0, \\
\nu_Q(i) > x & \implies i \in \text{En}(q) \implies S < p, q > (i) \neq 0.
\end{align*}
\]

This shows that \( \delta_Q S < p, q >= \delta_{st} < p, q > \), therefore \( S \in (\delta_{st} \rightarrow \delta_Q) \).
We conclude \( SEP 1 \subseteq (\delta_{st} \rightarrow \delta_Q) \), hence \( (\delta_{st} \rightarrow \delta_Q) \leq SEP 1 \). 
Q.E.D.

**Lemma 12**

Consider \( P, Q \) dense in \( \mathbb{R} \), \( P \cap Q = \emptyset, \nu_P : \mathbb{N} \rightarrow P, \nu_Q : \mathbb{N} \rightarrow Q \) bijective numberings, \( \delta_P \) and \( \delta_Q \) the derived weak separation representations of \( \mathbb{R} \). Then 

\[ SEP 1 \leq (\delta_P \rightarrow \delta_Q). \]

**Proof**

For \( p \in \Sigma^\omega, b \in \mathbb{N} \) let

\[ H(p, n) := \begin{cases} 0 & \text{if for no } k \ 01^k10 \text{ is a suffix of } p[n] \\ k & \text{if } 01^k10 \text{ is a suffix of } p[n]. \end{cases} \]

For \( p, q \in \Sigma^\omega \) define \( a_i, m_i, b_i, d_i \in \mathbb{N}, X_i \subseteq \mathbb{N} \) and a function \( \xi : \subseteq \mathbb{N} \rightarrow \mathbb{N} \) in steps \( i = 0, 1, \ldots \) as follows.

**Step 0:**

\[ m_0 := 0, d_0 := 0, X_0 := \emptyset, \xi(m_0) := 0. \]

Choose \( a_0, b_0 \) with

\[ \nu_Q(a_0) < \nu_Q(\xi(m_0)) < \nu_Q(b_0) \]

and

\[ \nu_Q(b_0) - \nu_Q(a_0) \leq 1. \]

**Step i, i = 2n + 1:**
- Case $H(p, n) = \emptyset$ or $H(p, n) \in X_{i-1}$:
  \[ m_i = m_{i-1}, d_i := d_{i-1}, X_i := X_{i-1}. \]
  Choose $a_i, b_i$ with
  \[ \nu_Q(a_{i-1}) < \nu_Q(a_i) < \nu_Q(m_i) < \nu_Q(b_i) < \nu_Q(b_{i-1}) \]
  and
  \[ \nu_Q(b_i) - \nu_Q(a_i) < 2^{-n}. \]

- Case $H(p, n) = k, k + m_{i-1}, k \notin X_{i-1}$:
  Define $m_i, d_i, a_i, b_i$ as in the case above and define additionally
  \[ X_i := X_{i-1} \cup \{k\}, \xi(k) := a_i \]

- Case $H(p, n) = m_{i-1} \notin X_i$:
  \[ X_i := X_{i-1} \cup \{m_{i-1}\}, \]
  \[ m_i := \min (\mathbb{N} \setminus X_i) \]
  \[ d_i := d_{i-1} + 1 \]
  Define $a_i, b_i, \xi(m_i)$ such that
  \[ \nu_Q \xi(m_{i-1}) < \nu_Q(a_i) < \nu_Q(m_i) < \nu_Q(b_i) < \nu_Q(b_{i-1}) \]
  and
  \[ \nu_Q(b_i) - \nu_Q(a_i) < 2^{-n} \]
  and
  \[ \nu_P(d_{i-1}) \notin [\nu_Q(a_i); \nu_Q(b_i)]. \]

Step $i$, $i = 2n + 2$:
(accordingly with $q$ instead of $i$, but to left of $\nu_Q \xi(m_{i-1})$ in the third case)

For all $i$, either $X_{i-1} = X_i$ and $m_{i-1} = m_i$ or $X_i \cup \{m_i\}$ extends $X_{i-1} \cup \{m_{i-1}\}$ by a single $j \in \mathbb{N}$ for which $\xi(j)$ is defined in Step $i$. Therefore $\xi$ is a well-defined function. By the construction

\[ \text{dom} (\xi) = En(p) \cup En(q) \cup \{\min (\mathbb{N} \setminus (En(p) \cup En(q)))\}, \]
and the function $f_1 : \Sigma^\omega \rightarrow \Sigma^\omega$ defined by

$$f_1 < p, q > (i) = \begin{cases} 
\xi(i) & \text{if } (\forall i) \xi(i) \text{ exists} \\
\text{div} & \text{otherwise}
\end{cases}$$

for all $p, q \in \Sigma^\omega$ is continuous. For any $p, q$ the sequence $[\nu_Q(a_i); \nu_Q(b_i)]$ converges to a number $x_{<p,q>} \in \mathbb{R}$. If $En(p) \cup En(q) \neq \mathbb{N}$, then the sequence $m_0, m_1, \ldots$ converges to $k := \min (\mathbb{N} \setminus (En(p) \cup En(q))),$ hence $x_{<p,q>} = \nu_Q(k) \in Q$. If $En(p) \cup En(q) = \mathbb{N}$ then Case 3 occurs for infinitely many $i$ and $\{d_0, d_1, \ldots\} \in \mathbb{N}$. For such a step $i$, $\nu_P(d_{i-1}) \notin [\nu_Q(a_i); \nu_Q(b_i)]$ hence $\nu_P(d_{i-1}) \neq x_{<p,q>}$. Therefore $(\forall j)[\nu_P(d_j) \neq x_{<p,q>}]$ i.e. $x_{<p,q>} \notin P$. Since $P \cap Q = \emptyset$, $x_{<p,q>} \notin P$ in any case.

Proposition: There is a continuous function $h : \Sigma^\omega \rightarrow \Sigma^\omega$ with

$$x_{<p,q>} = \delta_P f_2 < p, q > \text{ for all } p, q \in \Sigma^\omega.$$ 

Proof: For $p, q \in \Sigma^\omega$ and $k \in \mathbb{N}$ construct $h < p, q > (k)$ as follows. Find the smallest $i$ such that $\nu_P(k) \notin [\nu_Q(a_i); \nu_Q(b_i)]$. Since $x_{<p,q>} \notin P$ such a number $i$ exists. Define

$$h < p, q > (k) = \begin{cases} 
0 & \text{if } \nu_P(k) < \nu_Q(a_i) \\
1 & \text{otherwise}
\end{cases}$$

Obviously, $x_{<p,q>} = \delta_P h < p, q >$. The function $h$ is continuous. q.e.d.

The construction guarantees that $\nu_Q(\xi(k) < x_{<p,q>}$ if $k \in En(p)$ and $\nu_Q(\xi(k) > x_{<p,q>}$ if $k \in En(q)$. Now define

$$g(r, t)(k) := t(f_1(r))(k).$$

Then $g$ is continuous. We shall prove now that for any $T \in (\delta_p \rightarrow \delta_Q$ the function $S$ with $S(r) = g(r, Th(r))$ is a $1$–separator $S \in SEP 1$.

Let $r := < p, q >$ such that $En(p) \cup En(q) = \mathbb{N}$ and $\text{card}(\mathbb{N} \setminus (En(p) \cup En(q))) \leq 1$. Then $\xi = f_1 < p, q >$ exists and for all $i \in \mathbb{N} :$

$$i \in En(p) \quad \Rightarrow \quad \nu_Q(\xi(i) < x_{<p,q>}
\Rightarrow \quad \nu_Q f_1(r)(i) < \delta_P h(r)
\Rightarrow \quad \nu_Q f_1(r)(i) < \delta_Q Th(r)
\Rightarrow \quad Th(r)(f_1(r)(i)) = 0
\Rightarrow \quad g(r, Th(r))(i) = 0$$

20
and \( i \in En(q) \implies g(r, Th(r))(i) = 0 \) (correspondingly). Therefore, \( S \) is a 1-separator. This proves \( SEP_1 \leq (\delta_P \to \delta_Q) \). Q.E.D.

We combine the results as follows.

**Theorem 13**

Let \( P, Q \subseteq \mathbb{R} \) such that \( P \) and \( Q \setminus P \) are dense in \( \mathbb{R} \). Then

\[
SEP_1 \equiv (\delta_{st} \to \delta_P) \equiv (\delta_P \to \delta_Q)
\]

**Proof**

\[
SEP_1 \leq (\delta_P \to \delta_{Q \setminus P}) \quad \text{(Lemma 12)}
\]
\[
\leq (\delta_P \to \delta_Q) \quad \text{(since } \delta_Q \leq \delta_{Q \setminus P}, \text{ Theorem 7)}
\]
\[
\leq (\delta_{st} \to \delta_Q) \quad \text{(since } \delta_P \leq \delta_{st}, \text{ Theorem 7)}
\]
\[
\leq SEP_1 \quad \text{(Lemma 11)}
\]

Q.E.D.

Theorem 13 characterizes the degree of \( (\delta_P \leq \delta_Q) \) if \( Q \setminus P \) is dense in \( \mathbb{R} \). By Theorem 7, \( (\delta_P \leq \delta_Q) \) contains no continuous function, iff \( Q \setminus P \neq \emptyset \). Also for the special case that if \( Q \setminus P \) consists only of a single point we can give a characterization.

**Definition 14 (LLPO)**

**LLPO** is the set of all functions \( S : \subseteq \Sigma^\omega \to \Sigma^* \) with

\[
S < p, q > = \begin{cases} 
0 & \text{if } p \in 0^\omega \text{ and } q \neq 0^\omega \\
1 & \text{if } p \neq 0^\omega \text{ and } q \in 0^\omega \\
\in \{0, 1\} & \text{if } p = 0^\omega \text{ and } q = 0^\omega 
\end{cases}
\]

for all \( p, q \in \Sigma^\omega \).

21
No function \( S \in LLPO \) is continuous. The set \( LLPO \) corresponds to Brower's "lesser limited principle of omniscience" (Bridges and Richman, [1]).

**Theorem 15**

Let \( P, Q \) be dense in \( R \) with \( Q \setminus P \neq \emptyset \). Then:

1. \( LLPO \leq (\delta_P \rightarrow \delta_Q) \),
2. \( LLPO \equiv (\delta_P \rightarrow \delta_Q) \) if \( Q \setminus P \) has only one point.

**Proof:**

(1) Let \( \nu_P : \mathbb{N} \rightarrow P \) and \( \nu_Q : \mathbb{N} \rightarrow Q \) be arbitrary bijective numberings. There is some \( k \) with \( \nu_Q(k) \notin P \). Since \( P \) is dense, there is a sequence \([\nu_P(a_i); \nu_P(b_i)]\) of nested intervals converging to \( \nu_Q(k) \). Define a function \( f : \Sigma^\omega \rightarrow \Sigma^\omega \) by

\[
f < p, q > (j) := \begin{cases} 0 & \text{if } c_m = 0 \text{ and } \nu_P(j) < \nu_P(a_m) \\ 1 & \text{if } c_m = 0 \text{ and } \nu_P(j) \geq \nu_P(a_m) \\ 0 & \text{if } c_m = 1 \text{ and } \nu_P(j) < \nu_P(b_m) \\ 1 & \text{if } c_m = 1 \text{ and } \nu_P(j) \geq \nu_P(b_m) \\ 0 & \text{if } c_m = 2 \text{ and } \nu_P(j) < \nu_P(a_m) \\ 1 & \text{if } c_m = 2 \text{ and } \nu_P(j) \geq \nu_P(a_m) \end{cases}
\]

where

\[m := \mu_i[p(i) \neq 0 \text{ or } q(i) \neq 0 \text{ or } \nu_P(j) \notin [\nu_P(a_i); \nu_P(b_i)]]\]

and

\[c_m := \begin{cases} 0 & \text{if } p(m) = 0 \\ 1 & \text{if } p(m) = 0 \text{ and } q(m) = 0 \\ 2 & \text{otherwise.} \end{cases}\]

Then \( f \) is continuous.

Since the sequence of closed intervals \([\nu_P(Q_i); \nu_P(b_i)]\) converges to
\( \nu_Q(k) \notin P \), for some \( i \\ \nu_P(j) \notin [\nu_P(a_i); \nu_P(b_i)] \). Therefore \( m \) and \( c_m \) exist in any case, i.e. \( f < p, q > \) exists for all \( p, q \in \Sigma^\omega \). For \( p, q \in \Sigma^\omega \), define

\[
y(p, q) := \begin{cases} 
\nu_Q(k) & \text{if } p = q = 0^\omega \\
\nu_P(a_n) & \text{if } p(n) \neq 0 \\
\nu_P(b_n) & \text{if } p(n) = 0 \text{ and } q(n) \neq 0
\end{cases}
\]

where \( n = \mu_i[p(i) \neq 0 \text{ or } q(i) \neq 0] \). We show \( \delta_P f < p, q > = y(p, q) \) for all \( p, q \in \Sigma^\omega \).

Let \( p, q \in \Sigma^\omega \) and \( j \in \mathbb{N} \). If \( c_m = 0 \) then \( y(p, q) = \nu_P(a_m) \), if \( c_m = 1 \) then \( y(p, q) = \nu_P(b_m) \), and if \( c_m = 2 \) then \( \nu_P(a_m) < y(p, q) < \nu_P(b_m) \). In any of these cases one shows easily: \( \nu_P(i) < y(p, q) \implies f < p, q > (i) = 0 \) and \( \nu_P(i) > y(p, q) \implies f < p, q > (i) = 1 \). By Definition 6 we have \( y(p, q) = \delta_P f < p, q > \). Define \( h : \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega \) by \( h(r, s) := (0 \text{ if } s = 0, 1 \text{ otherwise}) \). \( h \) is continuous. Now consider \( T \in (\delta_P \rightarrow \delta_Q) \) and define \( S \) by \( S(r) := h(r, Tf(r)) \). Then \( S < p, q > \in \{0, 1\} \) for all \( p, q \in \Sigma^\omega \). If \( p = 0^\omega \) and \( q \neq 0^\omega \) then \( \delta_P f < p, q > = y(p, q) = \nu_P(b_n) > \nu_Q(k) \) for some \( n \), hence \( \delta_Q Tf < p, q > > \nu_Q(k) \) hence \( Tf < p, q > (k) = 0 \), hence \( S < p, q > (k) = 0 \). If \( p \neq 0^\omega \) and \( q = 0^\omega \) we obtain accordingly \( S < p, q > = 1 \).

This shows that \( S \in LLPO \).

(2) Assume \( Q \setminus P = \{\nu_Q(k)\} \). We have to show \( (\delta_P \rightarrow \delta_Q) \leq LLPO \).

Define \( f : \Sigma^\omega \rightarrow \Sigma^\omega \) by \( f(r) := < p, q > \) where

\[
p(n) := \begin{cases} 
0 & \text{if } r(n) = 0 \text{ or } \nu_Q(k) < \nu_P(n) \\
1 & \text{otherwise},
\end{cases}
\]

\[
q(n) := \begin{cases} 
0 & \text{if } r(n) \neq 0 \text{ or } \nu_Q(k) > \nu_P(n) \\
1 & \text{otherwise}.
\end{cases}
\]

Then \( f \) is continuous.

There is some function \( g : \mathbb{N} \rightarrow \mathbb{N} \) with \( \nu_Q(i) = \nu_P g(i) \) for all \( i \neq k \).

Define \( h : \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega \) by \( h(r, w)(k) := w \) and \( h(r, w)(i) := r g(i) \) if \( i \neq k \). Let \( S \) be some function from \( LLPO \). Define \( T \) by \( T(r) := h(r, S f(r)) \). Assume \( x = \delta_P(f) \). Then for \( i \neq k \), \( \nu_Q(i) < x \implies \nu_P g(i) < x \implies r g(i) = 0 \implies T(r)(i) = 0 \) and \( \nu_Q(i) > x \implies T(r)(i) \neq 0 \) (accordingly). At Position \( k \) we have:
\[ \nu_Q(k) < x \implies p = 0^\omega \text{ and } q \neq 0^\omega \implies S f(r) = S \cdot p, q \implies 0 \implies T(r)(k) = 0, \]
\[ \nu_Q(k) > x \implies q = 0^\omega \text{ and } p \neq 0^\omega \implies S f(r) = 0 \implies T(r)(k) = 0 \]
\[ \nu_Q(k) = x \implies p = q = 0^\omega \text{ and } S f(r) \in \{0, 1\} \text{ and } T(r)(k) \in \{0, 1\}. \]
Therefore we have \( T \in (\delta_P \implies \delta_Q) \) and since \( S \in LLPO \) was arbitrary we have \( (\delta_P \implies \delta_Q) \leq LLPO \).
Q.E.D.

A representation \( \delta : \subseteq \Sigma^\omega \rightarrow M \) is admissible, iff \( \delta' \leq \delta \) for all \( (\tau_c, \tau_s) \)-continuous representations \( \delta : \subseteq \Sigma^\omega \rightarrow M \), where \( \tau_c \) is the Cantor topology on \( \Sigma^\omega \) and \( \tau_s = \{ X \subseteq M \mid \delta^{-1} X \text{ open in } dom(\delta) \} \) is the final topology of \( \delta \) (Kreitz, Weihrauch [4, 10]). The admissible representations can be called “constructively effective”. The \( m \)-adic representations are not admissible (Kreitz, Weihrauch [10, 9]). We prove this for the broader class of weak separation representations.

**Theorem 16**

1. The final topology of any weak separation representation is the real line.

2. No weak separation representation is admissible.

**Proof:**

Let \( Q \subseteq \mathbb{R} \) be dense and let \( \delta \) be the derived weak separation representation. We show that the final topology of \( \delta \) is the real line topology \( \tau_\mathbb{R} \). We may assume \( r \in Dom(\delta) \implies range(r) \subseteq \{0, 1\} \).

\( \tau_\mathbb{R} \subseteq \tau_\delta \): The set \( \{(a; b) \in \mathbb{R} \mid a, b \in \mathbb{Q} \text{ and } a < b\} \) is a basis of \( \tau_\mathbb{R} \). If suffices to show that \( \delta^{-1}(a; b) \) is open in \( dom(\delta) \) for \( a, b \in Q, a < b \).

Let \( m, n \in \mathbb{N} \) with \( \nu(m) < \nu(n) \). Let

\[ D := \{ q \in \Sigma^\omega \mid \exists m', n' : \nu(m) < \nu(m') < \nu(n') < \nu(n) \text{ and } q(m') = 0 \text{ and } q(n') = 1 \} \]

Then \( D \) is open and \( p \in \delta^{-1}(\nu(m), \nu(n)) \iff p \in D \cap dom(\delta) \).
\( \tau_5 \subseteq \tau \): Let \( X \subseteq \mathbb{R} \) such that \( \delta^{-1}X \) is open in \( \text{dom}(\delta) \), i.e. \( X \in \tau_5 \). We have to show \( X \subseteq \tau \). Let \( \delta(p) = x \in X \). Then \( x \in \delta(w\Sigma^w) \subseteq X \) for some prefix \( w \) of \( p \).

Let \( w = p(0) \ldots p(n - 1) \).

Case \( x \neq \nu(i) \) for \( i < n \):

Let \( a := \max \{ \nu(i) \mid i < n, p(i) = 0 \} \), \( b := \min \{ \nu(i) \mid i < n, p(i) \neq 0 \} \).

Then \( x \in (a; b) = \delta(w\Sigma^w) \subseteq X \). This shows that \( x \) has an open neighbourhood in \( X \).

Case \( x = \nu(k) \) for some \( k < n \):

Case \( p(k) = 0 \): Let \( b := \min \{ \nu(i) \mid i < n, p(i) = 1 \} \), then \( [x; b] \subseteq \delta(w\Sigma^w) \subseteq X \). Let \( p'(k) := 1 \) and \( p'(i) := p(i) \) for \( i \neq k \). Then \( \delta(p') = x \) hence \( x \in \delta(w'\Sigma^w) \subseteq X \) for some prefix \( w' \) of \( p' \). As above we show \( [a; x] \subseteq \delta(w'\Sigma^w) \) for some \( a \in \mathbb{R} \). Therefore \( x \in (a; b) \subseteq X \), i.e. \( x \) has an open neighbourhood in \( X \).

Case \( p(k) = 1 \): (interchange \( p \) with \( p' \) in the above case).

Therefore the \( \delta \)-open set \( X \) is open in the real line.

(2) The standard representation \( \delta_{st} \) of \( \mathbb{R} \) is \( (\tau_c, \tau_m) \)-continuous, hence \( (\tau_c, \tau_e) \)-continuous. By Theorem 7, \( \delta_{st} \not\leq \delta \), hence \( \delta \) is not admissible.

Q.E.D.
5. Cut Representations, Iterated Fraction Representations

In Section 3 we have introduced the representation $\delta_\leq$. $\delta_\leq(p) = x$ iff $p$ enumerates the set of all $i$ such that $\nu_Q(i) < x$ ("$p$ enumerates the left cut of $x$"").

We shall now use characteristic functions of left cuts as names. For this Section 5 let $Q \subseteq \mathbb{IR}$ be some fixed subset dense in $\mathbb{IR}$ and let $\nu$ be some numbering of $Q$. As for weak separation representations the degree of a representation will depend in $Q$ but not on $\nu$.

Definition 17

Define representations $\gamma_\prec, \gamma_\succ$ and $\gamma_0$ of $\mathbb{IR}$ by:

\[
\begin{align*}
\gamma_\prec(p) = x & \iff p^{-1}\{1\} = \{i : \nu(i) < x\} \\
\gamma_\succ(p) = x & \iff p^{-1}\{1\} = \{i : \nu(i) > x\} \\
\gamma_0(p) = x & \iff \forall i : p_i(i) = \begin{cases} 0 & \text{if } \nu(i) < x \\ 1 & \text{if } \nu(i) = x \\ 2 & \text{if } \nu(i) > x \end{cases}
\end{align*}
\]

If $Q = \mathbb{Q}$ and $\nu$ is the standard numbering of $\mathbb{Q}$ then $\gamma_\prec$ is the left cut representation, $\gamma_\succ$ is the right cut representation, and $\gamma_0$ is (computationally equivalent to) the iterated fraction representation of $\mathbb{IR}$ (Deil [2]).

The following lemma can be proved easily (cf. Deil [2]).

Lemma 18

1. $\deg(\gamma_0)$ is the greatest lower bound of $\deg(\gamma_\prec)$ and $\deg(\gamma_\succ)$.

2. $\gamma_\prec \preceq \gamma_0$, $\gamma_\succ \preceq \gamma_0$, $\gamma_\prec \preceq \gamma_\succ$, $\gamma_\succ \preceq \gamma_\prec$.

3. If $\delta_Q$ is a weak separation representation then $\gamma_\prec \leq \delta_Q$ and $\gamma_\succ \leq \delta_Q$.

26
As in the former cases we introduce a new problem for characterizing the degree of translators for the cut representations.

Definition 19

Let $EC_1$ be the restriction of $EC$ to \{p \mid \text{card} \,(\mathbb{N} \setminus \text{En}(p)) \leq 1\}$, i.e.
\[
Cf \, EC_1(p) = \begin{cases} 
\text{En}(p) & \text{if } \text{card} \,(\mathbb{N} \setminus \text{En}(p)) \leq 1 \\
\text{div} & \text{otherwise.}
\end{cases}
\]

Lemma 20

\[(\delta_{st} \rightarrow \gamma_0) \leq \{EC_1\}\]

Proof:

By Lemma 2 we may use $\nu$ for determining $\delta_{st}$:

\[\delta_{st} < p, q > = x \iff \text{En}(p) = \{i \mid \nu(i) < x\} \text{ and } \text{En}(q) = \{i \mid \nu(i) > x\}.
\]

For $< p, q > \in \text{dom} \,(\delta_{st})$, $\text{En} < p, q > = \{i \mid \nu(i) = x\}$. Define $h : \subseteq \Sigma^\omega \times \Sigma^* \rightarrow \Sigma^*$ by

\[
h(< p, q >, 0^i) := \begin{cases} 
0 & \text{if } 01^{i+1}0 \text{ is a subword of } p[m] \\
2 & \text{if } 01^{i+1}0 \text{ is a subword of } q[m] \\
& \text{and not of } p[m]
\end{cases}
\]

if $m := \mu n[01^{i+1}1$ is a subword of $p[m]$ or of $q[m]$] exists. For all other arguments let $h(r, w) := \text{div}$. Then $h$ is continuous. Define $f : \subseteq \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma^\omega$ by

\[
f(< p, q >, r)(i) := \begin{cases} 
h(< p, q >, 0^i) & \text{if } r(i) \neq 0 \\
1 & \text{otherwise.}
\end{cases}
\]

Then $f$ is continuous and $T$ with $T < p, q > := f(< p, q >, EC_1 < p, q >)$ is a translator from $\delta_{st}$ to $\gamma_0$. (For $\delta_{st} < p, q > = x,$
\( \nu(i) < x \implies \ldots \implies T < p, q > (i) = 0, \nu(i) > x \implies T < p, q > (i) = 2, \nu(i) = x \implies T < p, q > (i) = 1). \)

Q.E.D.

**Lemma 21**

\( \{EC\ 1\} \leq (\gamma_\rightarrow \gamma_<) \)

**Proof**

First we assume that \( \nu : \mathbb{N} \rightarrow Q \) is bijective. For \( p \in \Sigma^\omega \) and \( n \in \mathbb{N} \) define

\[
Seg(p, n) := \max\{k \mid \forall j < k : 01^j1^0 \text{ is a subword of } p[n]\}.
\]

The sequence \( (Seg(p, n)) \) is unbounded if \( En(p) = \mathbb{N} \), otherwise it converges to \( \min (\mathbb{N} \setminus En(p)) \). For a given input \( p \in \Sigma^\omega \) we define \( a_n, d_n \in \mathbb{N} \) \((n = 0, 1, \ldots)\) as follows:

- \( n = 0 \)
- \( d_0 := 0 \); choose \( a_0 \) such that \( |\nu(a_0) - \nu(d_0)| > 1 \).
- \( n - 1 \rightarrow n \)
- Case \( Seg(p, n - 1) = Seg(p, n) \):
  - Then define \( d_n := d_{n-1}, a_n = a_{n-1} \)
- Case \( Seg(p, n - 1) < Seg(p, n) \):
  - Define \( d_n := d_{n-1} + 1 \);
  - if \( \nu(a_n) > \nu(a_{n-1}) + 2 \cdot 5^{-n} \) then \( a_n := a_{n-1} \), otherwise choose \( a_n \) such that
    \[
    \nu(a_{n-1}) + 3 \cdot 5^{-n} < \nu(a_n) < \nu(a_{n-1}) + 4 \cdot 5^{-n}.
    \]

The construction guarantees

\[
[\nu(a_n); \nu(a_n) + 5^{-n}] \subseteq [\nu(a_{n-1}); \nu(a_{n-1}) + 5^{-n-1}],
\]

especially the sequence \( (\nu(a_n))_{n \in \mathbb{N}} \) converges to some \( x_p \in \mathbb{R} \).

**Proposition 1**: For all \( k \):

\( (\exists n)k = a_n \) or \( (\exists n)\nu(k) \notin [\nu(a_n); \nu(a_n) + 5^{-n}] \)
Proof 1: Consider $k \in \mathbb{N}$.
Case $\mathcal{E}n(p) = \mathbb{N}$: In this case $\{d_0, d_1, \ldots\} = \mathbb{N}$, hence there is some smallest $n$ with $k = d_n$. The construction of $a_n$ guarantees $\nu(k) \notin [\nu(a_n); \nu(a_n) + 5^{-n}]$.
Case $\mathcal{E}n(p) \neq \mathbb{N}$: then there is some $m$ with $a_n = a_m$ for all $n \geq m$, thus $\nu(a_n) \rightarrow \nu(a_m)$. Assume $(\forall n)\nu(k) \in [\nu(a_n); \nu(a_n) + 5^{-n}]$. Then $\nu(a_n) \rightarrow \nu(k)$, hence $\nu(a_m) = \nu(k)$ and $k = a_m$.
q.e.d.(1)

Define $f : \Sigma^\omega \rightarrow \Sigma^\omega$ by

$$f(p)(k) := \begin{cases} 1 & \text{if } \nu(k) > \nu(a_m) \\ 0 & \text{otherwise} \end{cases}$$

where

$$m := \mu n[k = a_n \text{ or } \nu(k) \notin [\nu(a_n); \nu(a_n) + 5^{-n}]]$$

By Proposition 1, $f(p)(k)$ exists for all $p, k$; $f$ is continuous.

Proposition 2: $x_p := \lim_{n} \nu(a_n) = \gamma_{>f(p)}$

Proof (2): For $k \in \mathbb{N}$ let $m = \mu n[\ldots]$ (Def. of $f$)
Case $k = a_m : \nu(k) \leq x_p \iff \text{true} \iff f(p)(k) = 0$
Case $\nu(k) \notin [\nu(a_m); \nu(a_m) + 5^{-m}]$: Then

$$\nu(k) > x_p \iff \nu(k) > \nu(a_m) \iff f(p)(k) = 1$$

q.e.d.(2)

Define a function $g' : \subseteq \Sigma^\omega \times \Sigma^\omega \times \mathbb{N} \rightarrow \Sigma$ by

$$g'(p, s, k) := \begin{cases} \text{div} & \text{if } h(k) \text{ does not exist, else:} \\ 1 & \text{if } [01^{k+1}0 \text{ subword of } p[h(k)] \\ 0 & \text{otherwise} \end{cases}$$

where

$$h(k) := \mu n \quad \text{[01}^{k+1}0 \text{ subword of } p[n] \text{ or } (\text{Seg}(p, n) = k \text{ and } f(p)(a_n) \neq 1 \text{ and } s(a_n) \neq 1]}$$

and define $g : \subseteq \Sigma^\omega \times \Sigma^\omega \rightarrow \Sigma$ by

$$\text{dom}(g) = \{(p, s) | \forall k : g'(p, s, k) \text{ exists }\}$$

$$g(p, s)(k) := g'(p, s, k).$$
Then $g$ is continuous.

**Proposition 3:** For any $T \in (\gamma_\succ \to \gamma_\prec)$ we have $EC\,1(p) = g(p, T\,f(p))$ for all $p \in dom(\,EC\,1)$. 

**Proof 3:** For any $p \in \Sigma^\omega$ and $k \in \mathbb{N}$ we have 

$$k = Min(\mathbb{N} \setminus En(p)) 
\iff \begin{cases} 
(\exists n)[\text{Seg}(p, n) = k \text{ and } x_p = a_n] \\
(\exists n)[\text{Seg}(p, n) = k \text{ and } f(p)(a_n) \neq 1 \text{ and } Tf(p)(a_n) \neq 1].
\end{cases}$$

For each $p \in dom(\,EC\,1)$ and each $k \in \mathbb{N}$ we have either $(\exists n)0^{k+1}0$ is a subword of $p[n]$ or $k = Min(\mathbb{N} \setminus En(p))$. Therefore, if $p \in dom(\,EC\,1)$ and $s = Tf(p)$, then for each $k$ one and only one of the alternatives in the definition of $h(k)$ holds. We obtain for $p \in dom(\,EC\,1)$: 

$$g(p, Tf(p))(k) = 1 \iff k \in En(p).$$

q.e.d.(3)

**EC 1 $\leq (\gamma_\succ \to \gamma_\prec)$** follows from Proposition 3. Since the degrees of $\gamma_\succ$ and $\gamma_\prec$ do not depend on the particular numbering $\nu$ of $Q$, the assumption that $\nu$ is bijective has been made w.l.g.

Q.E.D.

**Theorem 22**

The following problems are equivalent ($\delta_Q$ is any weak separation representation based on $Q$).

$$\{EC\,1\}, (\delta_{st} \to \gamma_0), (\gamma_\prec \to \gamma_\succ), (\gamma_\succ \to \gamma_\prec), (\delta_Q \to \gamma_\succ),$$

$$\delta_Q \to \gamma_\prec), (\gamma_\prec \to \gamma_0), (\gamma_\succ \to \gamma_0), (\delta_{st} \to \gamma_\succ), (\delta_{st} \to \gamma_\prec), (\delta_Q \to \gamma_0)$$

**Proof:**

By Lemma 21, $\{EC\,1\} \leq (\gamma_\succ \to \gamma_\prec)$. Correspondingly $\{EC\,1\} \leq (\gamma_\prec \to \gamma_\succ)$ is proved. By Lemma 20. $(\delta_{st} \to \gamma_0), \leq \{EC\,1\}$. In general, if $\delta_1 \leq \delta_3$ and $\delta_4 \leq \delta_2$ then $(\delta_1 \to \delta_2) \leq (\delta_3 \to \delta_4)$. Using this property and the known reducibilities the above equivalences can be proved easily.

Q.E.D.

30
6. Final Remarks

Most of the translation problems considered in this paper are members of three degrees, the degree of

- \((En \rightarrow Cf) = \{EC\}\),
- \(SEP1\), and
- \(\{EC1\}\).

Easy arguments show

\[ SEP1 \leq \{EC1\} \leq \{EC\}. \]

M. Schröder (private communication) has proved \(EC1 < EC\), and presumably \(SEP1 < \{EC1\}\) can be shown.

For weak separation representations we have studied translations \((\delta_P \rightarrow \delta_Q)\) only for the case \(Q \setminus P\) dense and the case \(\text{card} (Q \setminus P) = 1\). The cases \(Q \setminus P = k \geq 1\), and \(Q \setminus P\) is infinite but non-dense are unsettled. The change of the dense set \(Q\) for cut-representations has not been investigated at all.
References

[1] Bridges, D.S.: Richman, F.:
Darstellungen und Berechenbarkeit reeller Zahlen, Informatik-Berichte Nr.51, Fernuni Hagen, 1984
[3] Kreitz, Ch.:
[4] Kreitz, Ch.; Weihrauch, K.:
[5] Mylatz, Uwe:
Vergleich unstetiger Funktionen in der Analysis, Diplomarbeit, Fernuniversität, Hagen, 1992
The Wadge degrees of the $F_{\alpha} \cup G_{\delta}$-subsets of the Cantor space (to appear)
Vergleich nicht konstruktiv lösbarer Probleme in der Analysis, Diplomarbeit, Fernuniversität, Hagen, 1989
[8] Wadge, W.:
Degrees of complexity of subsets of the Baire space, Notices of the AMS 1972, A - 714
[10] Weihrauch, Klaus:
Computability, Springer-Verlag, Berlin, Heidelberg, 1987
[11] Weihrauch, Klaus:
The lowest Wadge degrees of subsets of the Cantor space, Informatik-Berichte 107 Fernuniversität Hagen, 1991
[12] Weihrauch, Klaus: