# A Geometric Proof of a Formula for the Number of Young Tableaux of a Given Shape

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**Abstract**

This paper contains a short proof of a formula by Frame, Robinson, and Thrall [1] which counts the number of Young tableaux of a given shape. The proof is based on a simple but novel geometric way of expressing the area of a Ferrers diagram.

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# 1 Introduction

Let $\lambda = \{\lambda_1 \geq \lambda_2 \cdots \geq \lambda_m\}$ be a partition of $n$. The *Ferrers diagram* of $\lambda$ is an array of cells indexed by pairs $(i, j)$ with $1 \leq i \leq m$, $1 \leq j \leq \lambda_i$. A *Young tableau of shape* $\lambda$ (sometimes called a *standard tableau*) is an arrangement of the integers $1, 2, \ldots, n$ in the cells of the Ferrers diagram of $\lambda$ such that all rows and columns form increasing sequences. The total number of Young tableaux of shape $\lambda$ will be denoted $f(\lambda)$.

For each cell $(i, j)$ define the *hook* $H_{i,j}$ to be the collection of cells $(a, b)$ such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. Define the *hook length* $h_{i,j}$ to be the number of cells in $H_{i,j}$. (See Figure 1.)

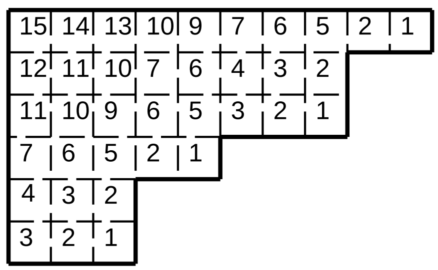


Figure 1: A Ferrers diagram with the hook lengths filled in

**Theorem 1** *(Frame-Robinson-Thrall [1]). If $\lambda$ is a partition of $n$, then*

$$f(\lambda) = \frac{n!}{\prod h_{i,j}},$$

*where the product is over all cells in the Ferrers diagram of $\lambda$.*

The first steps are the same as those found in [7] and [3] (see [5].) Define a function

$$F(\lambda) = \begin{cases} \frac{n!}{\prod h_{i,j}} & \text{if } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In any standard tableau, the integer $n$ must appear at a "corner", i.e., a cell which is at the end of some row and, simultaneously, at the end of a column. Removing this cell leaves a Young tableau of smaller shape. Thus the Frame-Robinson-Thrall formula follows by induction if it can be shown that

$$F(\lambda) = \sum_a F(\lambda_1, \lambda_2, \ldots, \lambda_{a-1}, \lambda_a - 1, \lambda_{a+1}, \ldots, \lambda_m). \tag{2}$$

(Note that the summation is, in effect, over all corners, since terms for which $\lambda_{a+1} > \lambda_a - 1$ are zero.)

For each corner $a$ in the diagram, let $I_a$ be the set of cells in the Ferrers diagram directly above $a$ and directly to the left of $a$, and define

$$\mathcal{G}_a(\lambda) = \prod_{b \in I_a} \frac{h_b}{h_b - 1}. \tag{3}$$

From equations (1), (2), and (3), the formula follows if it can be shown that

$$n = \sum_a \mathcal{G}_a(\lambda), \tag{4}$$

where the sum is over all corners $a$. Note that this can be interpreted as showing that the area $n$ of the Ferrers diagram is equal to the sum over all corners $a$ of $\mathcal{G}_a(\lambda)$. This is exactly what is proved in the next section.

## 2 A Geometric Proof of the Formula

Let $q$ be a positive integer and let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\beta = \{\beta_1, \beta_2, \ldots, \beta_q\}$ be sets of $q$ positive reals each. The *$q$-step staircase* of shape $(\alpha, \beta)$ is an area $A(\alpha, \beta)$ in the plane consisting of $\binom{q+1}{2}$ adjacent non-overlapping rectangles indexed by pairs $(i, j)$ with $1 \leq i \leq q$ and $1 \leq j \leq q - i + 1$. Rectangle $R_{i,j}$ has height $\alpha_i$, width $\beta_j$, and area $r_{i,j} = \alpha_i \cdot \beta_j$. The area of $A(\alpha, \beta)$ is $a(\alpha, \beta) = \sum_{i,j} r_{i,j}$. (See Figure 2.)

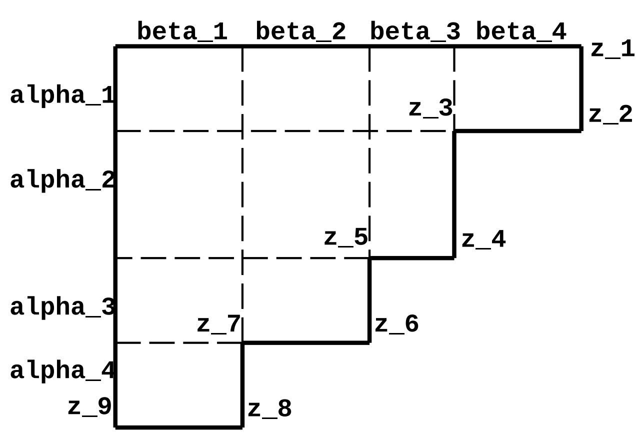


Figure 2: A staircase with 4 steps

Let $z_1, z_2, \ldots, z_{2q+1}$ be the locations of the "turning points" along the boundary of the staircase, indexed consecutively from the upper right-hand point to the bottom left-hand point. For all $i = 1, 2, \ldots, q$, define

$$G_i(\alpha, \beta) = \frac{\prod ||z_{2i} - z_{2j-1}||}{\prod ||z_{2i} - z_{2j}||}, \tag{5}$$

where $||\cdot||$ is the $\mathcal{L}_\infty$-norm, and where the sum in the numerator is over all $j = 1, 2, \ldots, q+1$ and the sum in the denominator is over all $j = 1, 2, \ldots, i-1, i+1, \ldots, q$. In words, the numerator of $G_i(\alpha, \beta)$ is the product of the Manhattan distances from point $z_{2i}$ to all odd indexed turning points, and the denominator of $G_i(\alpha, \beta)$ is the product of the Manhattan distances from $z_{2i}$ to all other even indexed turning points.

When we have an indexed set $r_1, \ldots$ of values, we use the shorthand $r_{[i,j]}$ to indicate $\sum_{k=i}^{j} r_k$. Similarly, when we have an indexed set $R_1, \ldots$ of sets, we use the shorthand $R_{[i,j]}$ to indicate $\cup_{k=i}^{j} R_k$. The main theorem of this section is the following.

**Theorem 2** *If $(\alpha, \beta)$ are positive reals that define a $q$-step staircase with area $a(\alpha, \beta)$, then $a(\alpha, \beta) = G_{[1,q]}(\alpha, \beta)$.*

PROOF: We first make the following simple but crucial observation, referring to Figure 3.

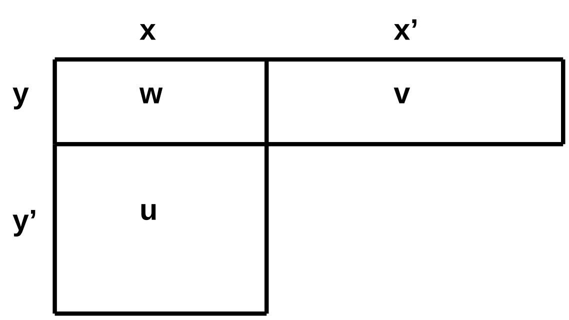


Figure 3: The basic geometric idea

Note that $u = xy'$, $v = x'y$, and $w = xy$. We can express $w$ in two equivalent ways as follows.

$$\begin{aligned} w &= \frac{y'}{x'+y'} \cdot w + \frac{x'}{x'+y'} \cdot w & (6) \\ &= \frac{y}{x'+y'} \cdot u + \frac{x}{x'+y'} \cdot v. & (7) \end{aligned}$$

Let $\psi_1, \ldots, \psi_q$ be a set of $q$ indeterminates. We associate a polynomial $p_{i,j}(\psi)$ as follows with each rectangle $R_{i,j}$ of the staircase. For all $i = 1, 2, \ldots, q$, let

$$p_{i,q-i+1}(\psi) = \alpha_i \cdot \beta_{q-i+1} \cdot \psi_i. \tag{8}$$

For all $(i,j)$ such that $2 \leq i+j < q+1$, let

$$\begin{aligned} p_{i,j}(\psi) &= \frac{\alpha_i}{\alpha_{[i+1,q-j+1]} + \beta_{[j+1,q-i+1]}} \cdot p_{[i+1,q-j+1],j}(\psi) \\ &+ \frac{\beta_j}{\alpha_{[i+1,q-j+1]} + \beta_{[j+1,q-i+1]}} \cdot p_{i,[j+1,q-i+1]}(\psi). \end{aligned} \tag{9}$$

From these definitions and the equivalence of equations (6) and (7) it is not hard to see that for all $(i,j)$,

$$p_{i,j}(1,1,\ldots,1) = \alpha_i \cdot \beta_j = r_{i,j}.$$

Let

$$p(\psi) = \sum_{i,j} p_{i,j}(\psi). \tag{10}$$

Then

$$a(\alpha,\beta) = \sum_{i,j} r_{i,j} = p(1,1,\ldots,1). \tag{11}$$

Let $C_i$ be the coefficient of $\psi_i$ in $p(\psi)$. We show that $C_i = G_i(\alpha,\beta)$, concluding the proof.

For all $(k,\ell)$ such that $1 \le k \le i$ and $1 \le \ell \le q-i+1$, let

$$G_i^{k,\ell} = \frac{\prod ||z_{2i} - z_{2j-1}||}{\prod ||z_{2i} - z_{2j}||}, \tag{12}$$

where the sum in the numerator is over all $j = k, \ldots, q-\ell+1$ and the sum in the denominator is over all $j = k, \ldots, i-1, i+1, \ldots, q-\ell+1$. Note that this is similar to the definition of $G_i(\alpha,\beta)$ in equation (5), except it is restricted to the substaircase that lies below and to the right of rectangle $R_{k,\ell}$. For all other $(k,\ell)$ where either $k > i$ or $\ell > q-i+1$, let $G_i^{k,\ell} = 0$. Define $C_i^{k,\ell}$ to be the coefficient of $\psi_i$ in $p_{k,\ell}(\psi)$. It is clear for all $(k,\ell)$ such that either $k > i$ or $\ell > q-i+1$ that $C_i^{k,\ell} = G_i^{k,\ell} = 0$. Furthermore, for all $i = 1, \ldots, q$, $C_i^{i,q-i+1} = G_i^{i,q-i+1} = \alpha_i \cdot \beta_{q-i+1}$. We prove by induction on the number of rectangles in the staircase that

$$C_i^{[k,i],[\ell,q-i+1]} = G_i^{k,\ell}, \tag{13}$$

and from this it immediately follows that $C_i = C_i^{[1,i],[1,q-i+1]} = G_i^{1,1} = G_i(\alpha,\beta)$. Consider $(k,\ell) \ne (i, q-i+1)$ such that $k \le i$ and $\ell \le q-i+1$. From equation (9),

$$\begin{aligned} C_i^{k,\ell} &= \frac{\alpha_k}{||z_{2(q-\ell+1)} - z_{2k}||} \cdot C_i^{[k+1,i],\ell} & (14)\\ &+ \frac{\beta_\ell}{||z_{2(q-\ell+1)} - z_{2k}||} \cdot C_i^{k,[\ell+1,q-i+1]}. & (15) \end{aligned}$$

If $k = i$ then, because $C_i^{[i+1,i],\ell} = 0$,

$$C_i^{i,[\ell,q-i+1]} = \left(1 + \frac{\beta_\ell}{||z_{2(q-\ell+1)} - z_{2i}||}\right) \cdot G_i^{i,\ell+1} = G_i^{i,\ell}. \tag{16}$$

Similarly, if $\ell = q-i+1$ then, because $C_i^{k,[q-i+2,q-i+1]} = 0$,

$$C_i^{[k,i],q-i+1} = \left(1 + \frac{\alpha_k}{||z_{2i} - z_{2k}||}\right) \cdot G_i^{k+1,q-i+1} = G_i^{k,q-i+1}. \tag{17}$$

When $k < i$ and $\ell < q-i+1$,

$$\begin{aligned} C_i^{[k+1,i],\ell} &= C_i^{[k+1,i],[\ell,q-i+1]} - C_i^{[k+1,i],[\ell+1,q-i+1]} \\ &= G_i^{k+1,\ell} - G_i^{k+1,\ell+1} = \frac{\beta_\ell}{||z_{2(q-\ell+1)} - z_{2i}||} \cdot G_i^{k+1,\ell+1}. & (18) \end{aligned}$$

Similarly,

$$C_i^{k,[\ell+1,q-i+1]} = \frac{\alpha_k}{||z_{2i} - z_{2k}||} \cdot G_i^{k+1,\ell+1}. \tag{19}$$

Then, since $||z_{2(q-\ell+1)} - z_{2k}|| = ||z_{2(q-\ell+1)} - z_{2i}|| + ||z_{2i} - z_{2k}||$, equations (14), (15), (18), and (19) show that

$$\begin{aligned} C_i^{[k,i],[\ell,q-i+1]} &= C_i^{k,\ell} + C_i^{[k+1,i],\ell} + C_i^{k,[\ell+1,q-i+1]} + C_i^{[k+1,i],[\ell+1,q-i+1]} \\ &= \left(1 + \frac{\beta_\ell}{||z_{2(q-\ell+1)} - z_{2i}||}\right) \cdot \left(1 + \frac{\alpha_k}{||z_{2i} - z_{2k}||}\right) \cdot G_i^{k+1,\ell+1} \\ &= G_i^{k,\ell}. \qquad (20) \end{aligned}$$

Finally, equations (16), (17), and (20) prove the theorem. ■

We observe that Theorem 2 can be used to directly prove the Frame-Thrall-Robinson formula. Let $\lambda$ be a partition of $n$ as before. The Ferrers diagram of $\lambda$ gives rise to a staircase $(\alpha, \beta)$ with area $n$ in an obvious way. Furthermore, it is not hard to verify that if corner cell $a$ in the Ferrer diagram is the $i^{th}$ corner point numbering from the upper right-hand corner to the bottom left-hand corner, then $\mathcal{G}_a(\lambda) = G_i(\alpha, \beta)$. Thus, Theorem 2 directly proves equation (4), and in turn this proves the formula.

## 3 A Probabilistic Viewpoint

We can use the results in the previous section to justify the "parachuting" algorithm of Greene-Nijenhuis-Wilf [3] for choosing a random Young tableau of a given shape. We explain the process with respect to a staircase of shape $(\alpha, \beta)$ with area $a(\alpha, \beta)$, and retain the notation of the previous section. Consider the following random process:

Choose an initial random point $t$ in the staircase uniformly at random.

Repeat until $t$ is in a corner rectangle $R_{i,q-i+1}$ for some $i$.

> Suppose that the current point $t$ is in rectangle $R_{ij}$.
>
> Choose $t$ uniformly from $R_{i,[j+1,q-i+1]} \bigcup R_{[i+1,q-j+1],j}$.

Starting at some point $t$ in $R_{i,j}$, the random choice of a point in the area directly to the right or below $R_{i,j}$ corresponds to expressing the area of $R_{i,j}$ according to equation (6). From the other equivalent way of expressing the area described in equation (7), and from (8), (9), (10), (11), and from the proof of Theorem 2 that shows that the coefficient of $\psi_i$ in the polynomial $p(\psi)$ is $G_i(\alpha, \beta)$ it follows that this random process ends in rectangle $R_{i,q-i+1}$ with probability $G_i(\alpha, \beta)/a(\alpha, \beta)$.

## 4 Historical Notes

The geometric proof of the formula described in Section 2 is based on [6], an unpublished manuscript written over 20 years ago by the author while still an undergraduate at M.I.T. as a research paper for a graduate level Combinatorics course taught by Richard Stanley.

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