# On the Runtime and Robustness of Randomized Broadcasting ⋆

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**Abstract**

In this paper, we study the following randomized broadcasting protocol. At some time $t$ an information $r$ is placed at one of the nodes of a graph. In the succeeding steps, each informed node chooses one neighbor, independently and uniformly at random, and informs this neighbor by sending a copy of $r$ to it. We begin by developing tight lower and upper bounds on the runtime of the algorithm described above. First, it is shown that on $\Delta$-regular graphs this algorithm requires at least $\log_{2-\frac{1}{\Delta}} n + \log_{(\frac{\Delta}{\Delta-1})^\Delta} n - o(\log n) \approx 1.69 \log_2 n$ rounds to inform all $n$ nodes. Together with a result of Pittel (On Spreading a Rumor, SIAM Journal on Applied Mathematics, 47(1):213-223) this bound implies that the algorithm has best performance on complete graphs among all regular graphs. For general graphs, we prove a slightly weaker lower bound of $\log_{2-\frac{1}{\Delta}} n + \log_4 n - o(\log n) \approx 1.5 \log_2 n$, where $\Delta$ denotes the maximum degree of $G$. We also prove two general upper bounds, $(1 + o(1))n \ln n$ and $\mathcal{O}(n\frac{\Delta}{\delta})$, respectively, where $\delta$ denotes the minimum degree.

The second part of this paper is devoted to the analysis of fault-tolerance. We show that if the informed nodes are allowed to fail in some step with probability $1-p$, then the broadcasting time increases by at most a factor $6/p$. As a by-product, we determine the performance of agent based broadcasting in certain graphs and obtain bounds for the runtime of randomized broadcasting on Cartesian products of graphs.

*Key words:* Randomized Algorithm, Distributed Computing

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## 1 Introduction

*Motivation and Related Work*

The study of information spreading in large networks has various fields of application in distributed computing. Consider for example the maintenance of replicated databases on name servers in a large network [5]. There are updates injected at various nodes, and these updates must be propagated to all the nodes in the network. In each step, a processor and its neighbor check whether their copies of the database agree, and if not, they perform the necessary updates. In order to be able to let all copies of the database converge to the same content, efficient and fault-tolerant broadcasting algorithms have to be developed.

Another well known example occurs in the analysis of epidemic disease. Often, mathematical studies about infection propagation make the assumption that an infected person spreads the infection equally likely to any member of a population [17], which leads to a complete graph for the underlying network. Whenever the question is, how fast the disease infects all persons, the problem reduces to the broadcasting problem. However, in most of these papers, spreaders are only active in a certain time window, and the question of interest is, whether on certain networks modeling personal contacts an epidemic outbreak occurs. Several threshold theorems involving the basic reproduction number, contact number, and the replacement number have been derived. See e.g. [14] for a collection of results concerning the mathematics of infectious diseases.

There is an enormous amount of experimental and theoretical study of broadcasting algorithms in various models and on different network topologies. Several (deterministic and randomized) algorithms have been developed and analyzed. In this paper we only concentrate on the efficiency of randomized broadcasting and mainly consider the runtime of the so called *push algorithm* [5] defined in the following way: In a graph $G = (V, E)$ of size $n := |V|$, we place at some time $t$ an information $r$ on one of the nodes. Then, in every succeeding time step, each *informed* vertex sends a copy of the information $r$ to one of its neighbors selected independently and uniformly at random.

The advantage of randomized broadcasting is in its inherent robustness against several kinds of failures and dynamical changes compared to deterministic schemes that either need substantially more time [10] or can tolerate only a relatively small number of faults [18]. Most papers dealing with randomized broadcasting analyze the runtime of the push algorithm in different graph classes. Pittel [20] proved that with a certain probability an information is spread to all nodes in a complete graph within $\log_2 n + \ln n + o(\log n)$ steps. Feige et al. [9] determined tight upper bounds of $\mathcal{O}(\Delta(\text{diam} + \log n))$ and $\mathcal{O}(n \log n)$, respectively, for general graphs, where $\Delta$ denotes the maximum degree of $G$. Furthermore it was shown that in random graphs and Hypercubes of size $n$, all nodes of the graph receive the information within $\mathcal{O}(\log n)$ steps, with high probability [1].

In [7] we considered the performance of the push algorithm in a Cayley graph known as the Star graph [1]. The $d$ dimensional Star graph $S_d$ has $n = d!$ vertices corresponding to the $d!$ permutations of $(1, 2, \ldots, d)$, and there is an edge between $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ iff an index $i \in \{2, \ldots, d\}$ exists such that $x_1 = y_i$, $x_i = y_1$, and $x_j = y_j$ for any $j \neq 1, i$. We have shown in [7] that in these graphs all nodes become informed within $\mathcal{O}(\log n) = \mathcal{O}(d \log d)$ steps by the push algorithm, w.h.p. This result was recently generalized in [8] to a class of Cayley graphs which also contains the Pancake and Transposition graph. For the $d$ dimensional Bubble Sort graph an asymptotically optimal upper bound of $\mathcal{O}(d^2)$ was established. Furthermore, we proved that the runtime of the push model is upper bounded by the mixing time of a certain random walk and an additional logarithmic factor on any graph.

A model related to the push algorithm has been introduced in [5] and is called *pull algorithm*. Here, any (informed or uninformed) node is allowed to call a randomly chosen neighbor, and the information is sent from the called to the calling node. Note that these kind of transmissions make only sense if new or updated informations occur frequently in the network so that every node places a random call in each round anyway.

It was observed in complete graphs of size $n$ that the push algorithm needs at least $\Omega(n \log n)$ transmissions to inform all nodes of the graph, w.h.p. However, in the case of the pull algorithm if a constant fraction of the nodes are informed, then within $\mathcal{O}(\log \log n)$ additional steps every node of this graph becomes informed as well, w.h.p. [5, 16]. This implies that in such graphs at most $\mathcal{O}(n \log \log n)$ transmissions are needed if the distribution of the information is stopped at the right time. Using this fact, Karp et al. [16] combined the push and pull algorithms, and introduced a termination mechanism to bound the number of total transmissions by $\mathcal{O}(n \log \log n)$ in complete graphs. Furthermore they showed that this result is asymptotically optimal among these kind of algorithms. They also considered communication failures and analyzed the performance of the algorithm in the case when the random connections established in each round follow an arbitrary probability distribution.

In [6], we introduced the so called *agent based broadcasting model*. In this model, at the beginning $n$ agents are distributed among the nodes and in each

[1] When we write "with high probability" or "w.h.p." we mean with probability at least $1 - n^{-1}$.

of the following steps, these agents jump from one node to another via edges chosen uniformly at random. An information $r$ placed initially on one node is carried by the agents to other vertices. If an agent visits an informed node, then the agent becomes informed, and any node visited by an informed agent becomes informed as well. It was shown that $\mathcal{O}(\log n)$ steps are sufficient to distribute $r$ among all nodes in random graphs. We also considered the performance of this model in bounded degree graphs and compared it to the behaviour of the push algorithm on different examples.

*Our Results*

We present a short overview of the most important new results of this paper and briefly discuss its relationship to previous results. All the following results refer to the push algorithm.

- We prove for $\Delta$-regular graph a lower bound of $\log_{2-\frac{1}{\Delta}} n + \log_{(\frac{\Delta}{\Delta-1})^{\Delta}} n - o(\log n)$. This is matched by the result of Pittel [20], which says that the runtime for complete graphs is $\log_2 n + \ln n \pm o(\log n)$.
- For non-regular graph, we show a slightly weaker lower bound of $\log_{2-\frac{1}{\Delta}} n + \log_4 n - o(\log n)$.
- For general graphs, we prove an upper bound of $(1+o(1))n \ln n$. This bound is matched by the graph $K_{n-1,1}$ and significantly improves over the upper bound of $12n \log n$ by Feige et al. [9].
- We consider the performance of broadcasting in presence of failures. If every vertex fails in some step with probability $1-p$ (independently of all other time steps, but *not* necessarily independently of all other vertices), then the broadcasting time increases by a factor of at most $\frac{6}{p}$.

*Organization*

The rest of the paper is organized in 5 sections. In the next section, we provide the basic notations required for the analysis of the randomized broadcasting algorithm. In Section 3 we develop lower bounds on the runtime of this algorithm, while Section 4 contains new upper bounds and a brief discussion of the worst-case ratio between deterministic and randomized broadcasting. We establish a robustness result in Section 5 which is followed by some applications. We conclude our paper in Section 6 by summarizing our results and pointing at some open problems.

## 2 Notations and Definitions

Throughout this paper, let $G = (V, E)$ be an unweighted, simple and connected graph of size $n := |V(G)|$ and diameter $\mathrm{diam}(G)$. We denote by $\delta$ and $\Delta$ the minimum and the maximum degree of $G$, respectively. Moreover let $N(v)$ be the set of neighbors for some $v \in V(G)$. As mentioned in the introduction, in this paper we mainly consider the following randomized broadcasting algorithm (also known as the push algorithm [5] or as randomized rumor spreading [20, 16]) which will be frequently written as $\mathcal{RBA}$. At time 0, one arbitrary node $s$ owns an information which is to be sent to all other nodes in $G$. In the following rounds $t = 1, 2, \ldots$, each *informed* node chooses one neighbor selected independently and uniformly at random, and forwards the information to this node (see also Figure 1 for a more formal description).

RandomizedBroadcastingAlgorithm (Push-Algorithm)

$t \leftarrow 0$
$I(t) \leftarrow \{s\}$
**while** $I(t) \neq V$ **do**
    $I(t+1) \leftarrow I(t)$
    **for** all nodes $u \in I(t)$ **do**
        $u$ chooses a neighbor $v$ uniformly at random
        $I(t+1) \leftarrow I(t+1) \cup \{v\}$
    **end for**
    $t \leftarrow t + 1$
**end while**

Fig. 1. Definition of the randomized broadcasting algorithm considered here

In this paper we focus on the number of rounds required to inform all nodes of $G$. Let $I(t)$ be the set of informed nodes at time $t$ and $H(t) := V \setminus I(t)$. Let $\mathsf{RT}(G, p) := \min\{t \in I\!N \mid \mathbf{Pr}[I(t) = V] \geq p\}$ be the runtime of $\mathcal{RBA}$ in $G$ (guaranteed with probability $p$), i.e., the number of rounds needed by $\mathcal{RBA}$ to inform all vertices of $G$ with a given probability $p$. Furthermore, let $\mathbf{E}[\mathsf{RT}(G)]$ denote the expected runtime of $\mathcal{RBA}$. We denote by $\mathsf{DT}(G) = \min\{t \in I\!N \mid \mathbf{Pr}[I(t) = V] > 0\}$ the runtime of a fastest deterministic broadcasting algorithm. Observe that on any graph $G$, $\mathsf{RT}(G, p) \geq \mathsf{DT}(G) \geq \max\{\mathrm{diam}(G), \log_2 n\}$, if $p > 0$.

## 3 Lower Bounds

There exists several techniques to prove lower bounds for deterministic broadcasting. In most cases these techniques make use of a bounded maximum degree which may lead to expressions using generalized Fibonacci-Numbers, see e.g. [3], or rely on the special structure of certain graphs [15]. Here, we will use probabilistic arguments. Before stating the first result recall that $\mathsf{RT}(G,p) \geq \alpha$ means that the probability that $\mathcal{RBA}$ informs all nodes of $G$ within $\alpha - 1$ steps is at most $p$.

**Theorem 1** *Let $G = (V,E)$ be an arbitrary $\Delta$-regular graph, where $\Delta \geq 2$. Then, for an arbitrary value $0 < p < 1$ we have*

$$\mathsf{RT}(G,p) \geq \log_{2-\frac{1}{\Delta}} n + \log_{(\frac{\Delta}{\Delta-1})^{\Delta}} n + \mathcal{O}(\log p).$$

*Proof.* In order to show the theorem we consider two cases. In the first case, we assume that $1 \leq |I(t)| \leq \frac{n}{4}$ and are going to prove that $\mathbf{E}[\,|I(t+1)|\,] \leq 2 \cdot (2 - \frac{1}{\Delta})^t$. Observe that on every graph $G$ with at least 2 nodes we have $|I(1)| = 2$. Furthermore, in each succeeding time step $t \geq 2$, every informed node has at least one informed neighbor. Therefore, an informed node chooses in round $t \geq 2$ some informed neighbor with probability at least $\frac{1}{\Delta}$ which implies that $\mathbf{E}[\,|I(t)| \mid |I(t-1)| = k\,] \leq (2 - \frac{1}{\Delta})k$. Thus we obtain for every $t \geq 2$ by using conditional expectations (see e.g. [19])

$$\begin{aligned}
\mathbf{E}[\,|I(t)|\,] &= \sum_{k=0}^{n} \mathbf{Pr}[\,|I(t-1)| = k\,] \cdot \mathbf{E}\left[\,|I(t)| \,\middle|\, |I(t-1)| = k\right] \\
&\leq \sum_{k=0}^{n} \mathbf{Pr}[\,|I(t-1)| = k\,] \cdot k \cdot \left(2 - \frac{1}{\Delta}\right) \\
&\leq \left(2 - \frac{1}{\Delta}\right) \cdot \mathbf{E}[\,|I(t-1)|\,].
\end{aligned}$$

Hence it holds for any $t \geq 1$ that $\mathbf{E}[\,|I(t)|\,] \leq 2 \cdot (2 - \frac{1}{\Delta})^{t-1}$. Now, Markov's inequality leads to

$$\begin{aligned}
\mathbf{Pr}\left[\left|I(\log_{2-\frac{1}{\Delta}}(p\frac{n}{4}))\right| \geq \frac{n}{4}\right] &\leq \frac{\mathbf{E}\left[\,|I(\log_{2-\frac{1}{\Delta}}(p\frac{n}{4}))|\,\right]}{\frac{n}{4}} \\
&\leq 2 \cdot \left(2 - \frac{1}{\Delta}\right)^{\log_{2-\frac{1}{\Delta}} p\frac{n}{4} - 1} \cdot \frac{4}{n} \\
&= \frac{2}{2 - \frac{1}{\Delta}} \cdot p\frac{n}{4} \cdot \frac{4}{n} \leq \frac{2}{\frac{3}{2}} p = \frac{4}{3} p.
\end{aligned}$$

Thus, we may conclude that with probability $1 - \frac{4}{3}p$ the number of informed nodes after $\log_{2-\frac{1}{\Delta}}(p\frac{n}{4}) = \log_{2-\frac{1}{\Delta}} n + \log_{2-\frac{1}{\Delta}} \frac{p}{4}$ rounds is at most $\frac{n}{4}$.

For the second case, let $t_0$ be the last time step when $|I(t)| \leq \frac{n}{4}$ (Note that $t_0$ is a random variable). In order to show that after step $t_0 + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np$ there is with probability $1 - \frac{4}{3}p - 2p$ at least one uninformed vertex, we consider the following procedure: In the steps $t_0+1, t_0+2, \ldots, t_0 + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np$ every (informed and uninformed) node chooses one neighbor, uniformly at random. Then, we show that all nodes of $V \setminus I(t_0)$ are chosen in at least one step by one of its neighbors with probability at most $2p$. Since an unchosen vertex of $H(t_0)$ remains necessarily uninformed after step $t_0 + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np$, we obtain that $|H(t_0 + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np)| \geq 1$ with probability at least $1 - \frac{4}{3}p - 2p$.

Consider some node $v \in V \setminus I(t_0)$. Then, in some step $v$ is chosen by none of its neighbors with probability $(\frac{\Delta-1}{\Delta})^\Delta$. Hence, a fixed vertex is not chosen in the time interval $(t_0, t_0 + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np]$ with probability

$$\left(\frac{\Delta-1}{\Delta}\right)^{\Delta \cdot \log_{(\frac{\Delta}{\Delta-1})^\Delta} np} = \frac{1}{np}.$$

For some vertex u, let $X_u = 1$ denote the event that in this time interval $u$ is always unchosen, and $X_u = 0$ otherwise. Then, for any $U \subseteq V \setminus (I(t_0) \cup u)$ it holds that

$$\mathbf{Pr}\left[ X_u = 1 \;\middle|\; \bigwedge_{u' \in U} X_{u'} = 0 \right] \geq \mathbf{Pr}\left[ X_u = 1 \right].$$

Consequently, we have

$$\begin{aligned} \mathbf{Pr}\left[ \wedge_{u \in H(t_0)} X_u = 0 \right] &\leq \left(1 - \frac{1}{np}\right)^{\frac{n}{2}} \\ &\leq e^{-\frac{1}{2p}} \leq 2p, \end{aligned}$$

as $x^x \geq e^{-1}$ for any $x \geq 0$.

Hence, by the Union bound [19] there exists at least one uninformed vertex at time step $t_0 + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np \geq \log_{2-\frac{1}{\Delta}} n + \log_{2-\frac{1}{\Delta}} \frac{p}{4} + \log_{(\frac{\Delta}{\Delta-1})^\Delta} np$ with probability $1 - \frac{4}{3}p - 2p$. □

The next lemma shows that the lower bound above is minimized for $\Delta = n-1$, if we neglect the $\mathcal{O}(\log p)$-term.

**Lemma 1** *The function*

$$f(x) := \log_{2-\frac{1}{x}} n + \log_{(\frac{x}{x-1})^x} n$$

*is strictly monotonously decreasing in $x \geq 2$ for any fixed $n \geq 1$.*

*Proof.* First $f$ can be rewritten by transforming the bases as

$$f(x) = \ln n \cdot \left( \frac{1}{\ln\left(2 - \frac{1}{x}\right)} + \frac{1}{x \ln \frac{x}{x-1}} \right).$$

The first derivate of $f$ can be written as $\frac{\partial f}{\partial x} := -\frac{A}{B}$, where

$$\begin{aligned} A :=& (x^2 - x)\ln^2\left(\frac{x}{x-1}\right) + 2\ln^2\left(2 - \frac{1}{x}\right) \cdot \ln\left(\frac{x}{x-1}\right) \cdot x^2 \\ &- 2\ln^2\left(2 - \frac{1}{x}\right) \cdot x - 3\ln^2\left(2 - \frac{1}{x}\right) \cdot \ln\left(\frac{x}{x-1}\right) \cdot x \\ &+ \ln^2\left(2 - \frac{1}{x}\right) \cdot \ln\left(\frac{x}{x-1}\right) + \ln^2\left(2 - \frac{1}{x}\right) \\ B :=& \ln^2\left(2 - \frac{1}{x}\right) \cdot x^2 \cdot (2x - 1) \cdot \ln^2\left(\frac{x}{x-1}\right) \cdot (x - 1). \end{aligned}$$

First observe that all factors of the denominator $B$ are positive for $x \geq 2$. Thus in order to prove the claim, it is enough to show that $A > 0$. First recall the inequalities based on the Taylor series of ln (cf. [2, p.5]):

$$\frac{1}{x-1} \geq \ln \frac{x}{x-1} \geq \frac{1}{x-1} - \frac{1}{2(x-1)^2}.$$

Replacing $\ln \frac{x-1}{x}$ in $A$ by the corresponding upper and lower bound and rearranging yields the following lower bound for $A$:

$$A \geq \frac{2 \cdot x^3 - \left(5 + 2\ln^2\left(2 - \frac{1}{x}\right)\right) \cdot x^2 + 3x - \ln^2\left(2 - \frac{1}{x}\right)}{2(x-1)^2}.$$

Using $x \geq 3$ and $\ln^2(2 - \frac{1}{x}) \leq \frac{1}{2}$ we obtain $A \geq \frac{6 \cdot x^2 - 6 \cdot x^2 + 3 \cdot x - 1/2}{2(x-1)^2} > 0$. The fact that $f(2) > f(3)$ can be verified numerically. □

**Theorem 2** *[20] On complete graphs $G = K_n$ we have that*

$$\mathsf{RT}(G, 1 - o(1)) \leq \log_2 n + \ln n + f(n),$$

*where $f(n) : \mathbb{N} \to \mathbb{N}$ is an arbitrary slow growing function, i.e.* $\lim_{n\to\infty} f(n) = \infty$.

Since this bound matches our lower bound for $\Delta = n-1$, we may conclude that complete graphs are best-case graphs among the class of all regular graphs.

Now we turn our attention to non-regular graphs.

**Theorem 3** *Let $G = (V, E)$ be an arbitrary graph. Then,*

$$\mathsf{RT}\Big(G, \frac{1}{\log n}\Big) \geq \log_{2-\frac{1}{\Delta}} n + \log_4 n - o(\log n).$$

*Proof.* In this proof we consider two cases. First, we assume that $\delta(G) \geq 2$. For a fixed node $v \in V$ let $M(v) := \sum_{u \in N(v)} 1/\deg(u)$. We call a node $v$ *good* if $M(v) \leq 1 + \gamma$ for some $\gamma = \Omega(1/\log\log n)$. Observe that $\sum_{v \in V} M(v) = n$. Choose a node $v \in V(G)$ uniformly at random. Then $\mathbf{E}[\, M(v) \,] \leq 1$ and hence Markov's inequality gives

$$\mathbf{Pr}[\, M(v) \geq 1 + \gamma \,] \leq \frac{1}{1+\gamma}.$$

Hence the number of good nodes is at least $(1 - \frac{1}{1+\gamma}) \cdot n$. Now, for any good node $v$ we have $M(v) = \sum_{w \in N(v)} 1/\deg(w) \leq 1 + \gamma$, where $\deg(w)$ is the degree of $w$. Consequently, node $v$ remains uninformed in some round with probability at least

$$\prod_{w \in N(v)} \Big(1 - \frac{1}{\deg(w)}\Big) = \prod_{w \in N(v)} \Big(1 - \frac{1}{\deg(w)}\Big)^{\frac{\deg(w)}{\deg(w)}} \geq \Big(\frac{1}{4}\Big)^{1+\gamma},$$

since $\Big(1 - \frac{1}{\deg(w)}\Big)^{\deg(w)} \geq \frac{1}{4}$ due to the fact that $(1 - \frac{1}{\deg w})^{\deg w}$ is monotonously increasing and $\deg(w) \geq 2$.

Now we concentrate on the case where $|I(t)| \leq \frac{1}{2}(1 - \frac{1}{1+\gamma})n$. As in the proof of Theorem 1 we obtain

$$\mathbf{Pr}\left[ \Big| I\Big(\log_{2-\frac{1}{\Delta}} \frac{p}{2}(1 - \frac{1}{1+\gamma})n\Big)\Big| \geq \frac{1}{2}(1 - \frac{1}{1+\gamma})n \right] \leq \frac{4}{3}p.$$

Thus by setting $p = \frac{1}{2\log n}$ we conclude that with probability $1 - \frac{2}{3\log n}$ the set of informed nodes after $\log_{2-\frac{1}{\Delta}}(\frac{1}{4\log n}(1 - \frac{1}{1+\gamma})n) = \log_{2-\frac{1}{\Delta}} n - \log_{2-\frac{1}{\Delta}} 4\log n - \log_{2-\frac{1}{\Delta}} \frac{\gamma+1}{\gamma} = \log_{2-\frac{1}{\Delta}} n - \mathcal{O}(\log\log n)$ rounds is at most $\frac{1}{2}(1 - \frac{1}{1+\gamma}) \cdot n$. As a consequence there are still $\frac{1}{2}(1 - \frac{1}{1+\gamma}) \cdot n$ good uninformed nodes left with this probability. Denote by $G(t)$ the set of good uninformed nodes at time step $t$ and let $t_0$ be the last time step such that $|G(t_0)| \geq \frac{1}{2}(1 - \frac{1}{1+\gamma}) \cdot n$ is satisfied. As in the proof of Theorem 1 we have that $\mathbf{E}[\, |G(t_0 + t)| \,] \geq (\frac{1}{4})^{(1+\gamma)t} \frac{1}{2}(1 - \frac{1}{1+\gamma}) \cdot n$.

Hence, after

$$\begin{aligned}\log_{4^{1+\gamma}} \frac{1}{2\log n}(1-\frac{1}{1+\gamma})n &= \log_{4^{1+\gamma}} n - \log_{4^{1+\gamma}} 2\log n - \log_{4^{1+\gamma}} \frac{\gamma+1}{\gamma} \\ &= (\log_{4^{1+\gamma}} 4)\cdot \log_4 n - \mathcal{O}(\log\log n) \\ &= \frac{1}{1+\gamma}\cdot \log_4 n - \mathcal{O}(\log\log n) \\ &= \log_4 n - \frac{\gamma}{1+\gamma}\log_4 n - \mathcal{O}(\log\log n) \\ &= \log_4 n - \mathcal{O}\left(\frac{\log n}{\log\log n}\right)\end{aligned}$$

additional time steps there exists at least one good uninformed node with probability $1-\mathcal{O}(\frac{1}{\log n})$. Hence, after $\log_{2-\frac{1}{\Delta}} n + \log_4 n - \mathcal{O}(\log\log n)$ rounds there still exists some (good) uninformed node with probability $1-\frac{1}{\log n}$.

Let us now assume that the minimal degree in $G$ equals 1, and consider two further cases. First, let the number of nodes of degree 1 be at least $\frac{2n}{\log\log n}$. Then, using the same techniques as in the first case, we can show that with probability $1-\frac{2}{3\log n}$ we need more than $\log_{2-\frac{1}{\Delta}} n - o(\log n)$ rounds to inform at least $\frac{n}{\log\log n}$ nodes. Now, if there are at most $\frac{n}{\log\log n}$ nodes informed in $G$, then at least $\frac{n}{\log\log n}$ nodes of degree 1 are still uninformed. Since a node of degree 1 is contacted in one step with probability at most $1/2$, we can use the arguments of the first case to show that with probability $1-\mathcal{O}(\frac{1}{\log n})$ we need even $\log_2 n - o(\log n)$ additional time steps to inform all nodes of the graph.

Now, we consider the case when the number of nodes of degree 1 is less than $\frac{2n}{\log\log n}$. By setting $\gamma = \frac{(6+\Omega(1))}{\log\log n}$, we obtain that there are more than $\frac{6n}{\log\log n}$ good nodes in $G$. Therefore, at least $\frac{4n}{\log\log n}$ good nodes do not have any neighbor of degree 1. Consequently, $\frac{2n}{\log\log n}$ of these nodes have degree at least 2.

As in the first case, we need with probability at least $1-\frac{2}{3\log n}$ more than $\log_{2-\frac{1}{\Delta}} n - o(\log n)$ rounds to inform at least $\frac{n}{\log\log n}$ nodes. Now, if there are at most $\frac{n}{\log\log n}$ informed nodes in $G$, then at least $\frac{n}{\log\log n}$ good nodes of degree 2, which are not adjacent to any node of degree 1, are still uninformed. Thus, we can use the same arguments as in the first case, and obtain that with probability $1-\mathcal{O}(\frac{1}{\log n})$ we need at least $\log_4 n - o(\log n)$ additional steps to inform all these nodes in the graph. □

## 4 Upper Bounds and Applications

### 4.1 Upper Bounds

First, we consider graphs with $\frac{\Delta}{\delta} \leq \log n$ and improve the upper bound $\mathcal{O}(n \log n)$ stated in [9] for general graphs.

**Proposition 1** *For any graph $G = (V, E)$ it holds that $\mathbf{E}[\,\mathsf{RT}(G)\,] = \mathcal{O}(\frac{\Delta}{\delta} \cdot n)$.*

*Proof.* We consider two cases concerning $|E(I(t), H(t))|$, where $E(I(t), H(t))$ denotes the set of edges between $I(t)$ and $H(t)$ at time step $t$.

(1) $|E(I(t), H(t))| \leq \frac{\delta}{4}$: Since $G$ is connected, after $t_0 = \mathcal{O}(\Delta)$ expected rounds it holds that $I(t + t_0) \supseteq I(t) \cup \{u\}$ with $u \in H(t)$, where we may assume w.l.o.g. that $I(t + t_0) = I(t) \cup \{u\}$. Consequently we have at time step $t + t_0$ that $|E(\{u\}, H(t) \backslash \{u\})| \geq \frac{3\delta}{4}$. Hence, $\Theta(\delta)$ neighbors of $u$ will be informed by $u$ directly within the following $\mathcal{O}(\delta)$ rounds.
(2) $|E(I(t), H(t))| \geq \frac{\delta}{4}$: In this case after $\mathcal{O}(\frac{\Delta}{\delta})$ expected rounds another node becomes informed.

Thus, in either case the ratio between the expected time and the number of newly informed vertices is $\mathcal{O}(\frac{\Delta}{\delta})$ and the claim follows. □

The proposition above implies that the push algorithm has a runtime of $\mathcal{O}(n)$ for any regular graph. However, we now give a construction of a $\Delta$-regular graph $G$ whose expected runtime is $\Omega(n)$. Assume w.l.o.g. that $2 \leq \Delta \leq n - 1$ divides $n$ and consider $n/\Delta$ complete graphs $K_\Delta$, which are denoted by $G_0, \ldots, G_{n/\Delta - 1}$. Now arrange the complete graphs in a cycle and connect two neighboring complete graphs by two node-disjoint edges $\{u_i, u_{(i+1) \mod (n/\Delta)}\}$, $\{v_i, v_{(i+1) \mod (n/\Delta)}\}$, where $u_i, v_i \in G_i$ and $u_{(i+1) \mod (n/\Delta)}, v_{(i+1) \mod (n/\Delta)} \in G_{(i+1) \mod (n/\Delta)}$. Finally, we make the graph regular by removing the edges $\{u_i, v_i\}$ in $G_i$ and $\{u_{(i+1) \mod (n/\Delta)}, v_{(i+1) \mod (n/\Delta)}\}$ in $G_{(i+1) \mod (n/\Delta)}$. Note that the resulting graph is a $\Delta - 1$ regular graph and it is easy to see that the push algorithm requires $\Omega(n)$ rounds in expectation to inform all nodes of the resulting graph, as it takes $\Omega(\frac{n}{\Delta})$ steps for the information to be propagated from one complete graph to a neighboring complete graph.

We will now focus on the non-regular case. In [9] it is shown that for any graph $G$ it holds that $\mathsf{RT}(G, 1 - n^{-1}) \leq 12 n \ln n$. The following theorem reduces the constant from 12 to $1 + o(1)$ while still guaranteeing this bound with a probability tending to one.

Interestingly, it is easy to see that the graph $K_{1,n-1}$ matches this bound. Before stating the theorem, we have to introduce some further notation and list three technical lemmas required for the proof.

We denote by $\mathsf{Exp}(\lambda)$ the exponential distribution with parameter $\lambda > 0$ (and thus with mean $\frac{1}{\lambda}$) and by $\mathsf{Geo}(p)$ the geometric distribution with parameter $p > 0$ (and mean $\frac{1}{p}$) (cf. [19]).

**Definition 1** *Let $X$ and $Y$ be two probability distributions on $\mathbb{R}$. We say that $X$ stochastically dominates $Y$ if*

$$\mathbf{Pr}\left[ X \geq k \right] \leq \mathbf{Pr}\left[ Y \geq k \right]$$

*for any $k \in \mathbb{R}$.*

**Lemma 2 ([9])** *Let $(u_0, u_1, \ldots, u_l)$ be any shortest path in $G$ from $u_0$ to $u_l$. Then,*

$$\sum_{k=0}^{l} \deg(u_k) \leq 3n.$$

**Lemma 3** *Let $Y := \sum_{i=1}^{k} \mathsf{Geo}(\frac{1}{x_i})$, such that $\sum_{i=1}^{k} x_i = x$ and $\forall i \in \{1, \ldots, k\}$ : $2 \leq x_i$. Then we have*

$$\mathbf{Pr}\left[ Y = y \right] \leq (y + k - 1)^{k-1} \cdot e^{-\frac{y-k}{x}} \prod_{i=1}^{k} \left( \frac{1}{x_i} \right).$$

*Proof.* We have

$$\begin{aligned}
\mathbf{Pr}\left[\, Y = y \,\right] &= \sum_{\substack{1 \le \alpha_i \le y-k+1 \\ \sum_{i=1}^k \alpha_i = y}} \prod_{i=1}^{k} \left( \left(1 - \frac{1}{x_i}\right)^{\alpha_i - 1} \frac{1}{x_i} \right) \\
&\le \prod_{i=1}^{k} \left(\frac{1}{x_i}\right) \sum_{\substack{1 \le \alpha_i \le y-k+1 \\ \sum_{i=1}^k \alpha_i = y}} \prod_{i=1}^{k} \left(\frac{1}{e}\right)^{\frac{\alpha_i - 1}{x_i}} \\
&= \prod_{i=1}^{k} \left(\frac{1}{x_i}\right) \sum_{\substack{1 \le \alpha_i \le y-k+1 \\ \sum_{i=1}^k \alpha_i = y}} \left(\frac{1}{e}\right)^{\sum_{i=1}^k \frac{\alpha_i - 1}{x_i}} \\
&\le \prod_{i=1}^{k} \left(\frac{1}{x_i}\right) \sum_{\substack{1 \le \alpha_i \le y-k+1 \\ \sum_{i=1}^k \alpha_i = y}} \left(\frac{1}{e}\right)^{\frac{\sum_{i=1}^k (\alpha_i - 1)}{\sum_{i=1}^k x_i}} \\
&\le \prod_{i=1}^{k} \left(\frac{1}{x_i}\right) \sum_{\substack{0 \le \alpha_i \le y \\ \sum_{i=1}^k \alpha_i = y}} \left(\frac{1}{e}\right)^{\frac{y-k}{x}} \\
&= \prod_{i=1}^{k} \left(\frac{1}{x_i}\right) \binom{y+k-1}{k-1} \left(\frac{1}{e}\right)^{\frac{y-k}{x}} .
\end{aligned}$$

□

**Lemma 4** *The distribution* $\mathsf{Geo}(\frac{1}{\deg u_i})$ *stochastically dominates the distribution* $\mathsf{Exp}(\frac{1}{\deg u_i}) + 1$.

*Proof.* Consider the two random variables $X \sim \mathsf{Geo}(\frac{1}{\deg u_i})$ and $Y \sim \mathsf{Exp}(\frac{1}{\deg u_i}) + 1$. Then

$$\mathbf{Pr}\left[\, X \ge k \,\right] = \left(1 - \frac{1}{\deg u_i}\right)^{k-1} \le e^{-\frac{k-1}{\deg u_i}} = \mathbf{Pr}\left[\, Y \ge k \,\right].$$

□

**Theorem 4** *For any graph* $G = (V, E)$ *it holds that*

$$\mathsf{RT}(G, 1 - o(1)) \le (1 + o(1)) \cdot n \ln n.$$

*Proof.* Let $u_0$ be initially informed and $P := (u_0, u_1, \ldots, u_l)$ be a shortest path from $u_0$ to $u_l$. Let us define

$$A := \{u_i \in P \mid \deg(u_i) > \gamma n\},\ B := \{u_i \in P \mid \deg(u_i) \le \gamma n\},$$

where $\gamma := \frac{1}{\ln\ln n}$. We first consider the number of rounds $X_B$ required for the information to proceed from each node of $B$ to its corresponding successor on $P$. Denote by $U_i$ the number of time steps needed for the information to be sent from node $u_i$ to $u_{i+1}$ directly. Then, $U_i$ is geometrically distributed with parameter $\frac{1}{\deg(u_i)}$. Due to Lemma 4 we may estimate this distribution by $\mathsf{Exp}(\frac{1}{\deg(u_i)}) + 1$. Notice that the moment-generating function of $\mathsf{Exp}(\frac{1}{\deg u_i})$ is given by

$$\mathbf{E}\left[e^{tU_i}\right] = \frac{\frac{1}{\deg u_i}}{\frac{1}{\deg u_i} - t},$$

where $t < \frac{1}{\deg u_i}$. Now use the Chernoff-Bound for the moment-generating function [19], and recall that $\sum_{v\in B} \deg_v \leq 3n$ by Lemma 2, to obtain

$$\begin{aligned}
\mathbf{Pr}\left[X_B \geq y + |B|\right] \leq \frac{\mathbf{E}\left[e^{tX_B}\right]}{e^{ty}} &= \frac{\prod_{i:u_i\in B}\left(\frac{\frac{1}{\deg u_i}}{\frac{1}{\deg u_i} - t}\right)}{e^{ty}} \\
&\overset{t=\frac{1}{2\gamma n}}{=} \frac{\prod_{i:u_i\in B}\left(\frac{\frac{1}{\deg u_i}}{\frac{1}{\deg u_i} - \frac{1}{2\gamma n}}\right)}{e^{\frac{1}{2\gamma n}y}} \\
&= \frac{\prod_{i:u_i\in B}\left(\frac{1}{1-\frac{\deg u_i}{2\gamma n}}\right)}{e^{\frac{1}{2\gamma n}y}},
\end{aligned}$$

where the $+|B|$ term comes from the fact that we have replaced the $\mathsf{Geo}(\frac{1}{\deg(u_i)})$ random variables by $\mathsf{Exp}(\frac{1}{\deg(u_i)}) + 1$. Since $(\frac{1}{1-\frac{1}{x}})^x$ is monotonously decreasing in $x$ and $\deg u_i \leq \gamma n$ because of $u_i \in B$ we get

$$\begin{aligned}
\mathbf{Pr}\left[X_B \geq y + |B|\right] &\leq \frac{\prod_{i:u_i\in B} 4^{\frac{\deg u_i}{2\gamma n}}}{e^{\frac{1}{2\gamma n}y}} \\
&= \frac{4^{\sum_{i:u_i\in B}\frac{\deg u_i}{2\gamma n}}}{e^{\frac{1}{2\gamma n}y}} \leq \frac{4^{\frac{3}{2\gamma}}}{e^{\frac{1}{2\gamma n}y}}.
\end{aligned}$$

Then, for $y = 4\gamma \cdot n \cdot \ln n$ we finally obtain

$$\mathbf{Pr}\left[X_B \geq \frac{4}{\ln\ln n}\cdot n\cdot \ln n + n\right] \leq \frac{4^{\frac{3}{2}\ln\ln n}}{e^{2\ln n}} = \frac{4^{\frac{3}{2}\ln\ln n}}{n^2} = \tilde{\mathcal{O}}(n^{-2}),$$

where $\tilde{\mathcal{O}}$ suppresses all polylogarithmic factors in $n$, e.g. $n^2\log^4 n = \tilde{\mathcal{O}}(n^2)$.

For the second part we have to consider the nodes of the set $A$. Observe that any node not lying on this shortest path can only be adjacent to at most three consecutive nodes [9]. Now fix any node $u_i \in A$. If any successor $u_{i+1}$ or $u_{i+2}$ shares more than $n^{2/3}$ common neighbors with $u_i$, then the expected time to inform $u_{i+1}$ (or $u_{i+2}$) is less than $2n^{5/6}$. This can be shown by using the fact that the expected time to inform $n^{1/2}$ of these neighbors is at most $\sum_{k=1}^{n^{1/2}} \frac{\deg(u_i)}{n^{2/3}-k} = \mathcal{O}(n^{1/2} \cdot n^{1/3}) \leq \frac{3}{2} n^{5/6}$. Having informed $\sqrt{n}$ of these common neighbors, $u_{i+1}$ (or $u_{i+2}$) becomes informed in one of the succeeding rounds with probability at least

$$1 - \left(1 - \frac{1}{n}\right)^{\sqrt{n}} \geq 1 - \left(\frac{1}{e}\right)^{1/\sqrt{n}},$$

and since $e^x \geq x + 1$, this probability is at least

$$\geq 1 - \frac{1}{1 + \frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n} + 1}.$$

Hence, after further expected $\sqrt{n}+1$ rounds, $u_{i+1}$ (or $u_{i+2}$) becomes informed. Unfortunately, this situation with these many common neighbors has also some drawback; when $u_{i+1}$ (or $u_{i+2}$) are supposed to propagate the information further, this vertex may be distracted by the large set of common neighbors with $u_i$. We therefore require some more detailed analysis.

Let $N(u_i, u_{i+1})$ denote the set of common neighbors of $u_i$ and $u_{i+1}$. In order to simplify the notation we write $u \sim v$ for two nodes $u$ and $v$, if $|N(u, v)| \geq n^{2/3}$. In this case, it might sometimes be helpful to imagine this set as some supernode adjacent to $u_i$ and $u_{i+1}$ with multiple edges. We denote by $S(u_i, u_{i+1})$ this supernode.

In order to benefit from the detours via supernodes, we describe now a transformation of the original path $P$ (which is an arbitrary, but fixed shortest path) to another path $P'$. As $P$, $P'$ starts with the node $u_0$. Assume that we have constructed the path $P' = (v_0 = u_0, v_1, \ldots, v_j)$ till some vertex $v_j = u_i$ lying on $P$. We distinguish now between three cases on how to extend $P'$ further.

(1) $|N(u_i, u_{i+2})| \geq n^{2/3}$: Then we extend $P'$ by the supernode $S(u_i, u_{i+2})$ and $u_{i+2}$, i.e. $P' = (v_0 = u_0, v_1, \ldots, v_j, v_{j+1} = S(u_i, u_{i+2}), v_{j+2} = u_{i+2})$.
(2) $|N(u_i, u_{i+2})| < n^{2/3} \wedge |N(u_i, u_{i+1})| \geq n^{2/3}$: In this case, we extend $P'$ by the supernode $S(u_i, u_{i+1})$ and $u_{i+1}$,
(3) $|N(u_i, u_{i+2})| < n^{2/3} \wedge |N(u_i, u_{i+1})| < n^{2/3}$: In this case, we extend $P'$ just by $u_{i+1}$ as in the original path $P$.

Let us first consider the subset of vertices $A'' \subseteq A$ which are followed in $P'$ by a supernode. Let $X_{A''}$ be the sum over all times it requires for the information to proceed from $v_i \in A''$ via some supernode $S(v_i, v_{i+1})$ to $v_{i+1}$. As $|A| \leq \log\log n$, we have $\mathbf{E}\left[\, X_{A''} \,\right] \leq 2n^{5/6} \cdot \log\log n$. Thus, the probability that this process takes more than $4n^{5/6} \ln\ln n$ steps is at most $\frac{1}{2}$. Due to independence, we may simply iterate and obtain

$$\mathbf{Pr}\left[\, X_{A''} \geq 4n \ln\ln n \,\right] \leq 2^{-n^{1/6}} < n^{-2}.$$

Hence, the only remaining nodes on $P'$ to consider are in $A$ and their successors on $P'$ are no supernodes. Let $A'$ be this subset of nodes and let $|N'(u_i)|$ denote the neighbors of $u_i$ which are only adjacent to $u_i$ on $P$. By definition of $A'$

$$\begin{aligned}
\sum_{i:u_i \in A'} \deg(u_i) &= \sum_{i:u_i \in A'} |N(u_i,u_{i-2}) \cup N(u_i,u_{i-1}) \cup N'(u_i) \cup N(u_i,u_{i+1}) \cup N(u_i,u_{i+2})| \\
&\leq \sum_{i:u_i \in A'} |N(u_i,u_{i-2}) \cup N(u_i,u_{i-1})| + |N'(u_i)| + n^{2/3} + n^{2/3} \\
&\leq \sum_{i:u_i \in A'} (|N(u_i,u_{i-2}) \cup N(u_i,u_{i-1})|) + \sum_{i:u_i \in A'} |N'(u_i)| + \\
&\quad \mathcal{O}(n^{2/3} \log\log n) \\
&\leq \sum_{\substack{i:u_i \in A' \\ u_{i-2} \not\sim u_i, u_{i-1} \not\sim u_i}} (|N(u_i,u_{i-2})| + |N(u_i,u_{i-1})|) + \\
&\quad \sum_{\substack{i:u_i \in A' \\ u_{i-2} \not\sim u_i, u_{i-1} \sim u_i}} (|N(u_i,u_{i-2})| + |N(u_i,u_{i-1})|) + \\
&\quad \sum_{\substack{i:u_i \in A' \\ u_{i-2} \sim u_i, u_{i-1} \not\sim u_i}} (|N(u_i,u_{i-2}) \cup N(u_i,u_{i-1})|) + \\
&\quad \sum_{\substack{i:u_i \in A' \\ u_{i-2} \sim u_i, u_{i-1} \sim u_i}} (|N(u_i,u_{i-2}) \cup N(u_i,u_{i-1})|) + \\
&\quad \sum_{i:u_i \in A'} |N'(u_i)| + \mathcal{O}(n^{2/3} \log\log n),
\end{aligned}$$

and since $|N(u_{i-1},u_{i-2}) \cap N(u_i,u_{i-2})| \geq n^{2/3}$ would imply $u_{i-1} \sim u_i$,

$$\begin{aligned}
\sum_{i:u_i \in A'} \deg(u_i) \leq &\sum_{\substack{i:u_i \in A' \\ u_{i-2} \not\sim u_i, u_{i-1} \not\sim u_i}} (|N(u_i,u_{i-2})| + |N(u_i,u_{i-1})|) + \\
&\sum_{\substack{i:u_i \in A' \\ u_{i-2} \not\sim u_i, u_{i-1} \sim u_i}} (|N(u_i,u_{i-2})| + |N(u_i,u_{i-1})|) + \\
&\sum_{\substack{i:u_i \in A' \\ u_{i-2} \sim u_i, u_{i-1} \not\sim u_i}} (|N(u_i,u_{i-2}) \backslash N(u_{i-1},u_{i-2})|) + \\
&\sum_{\substack{i:u_i \in A' \\ u_{i-2} \sim u_i, u_{i-1} \sim u_i}} (|N(u_i,u_{i-2}) \cup N(u_i,u_{i-1})|) + \\
&\sum_{i:u_i \in A'} |N'(u_i)| + \mathcal{O}(n^{2/3} \log\log n)
\end{aligned}$$

$$= \underbrace{\sum_{\substack{i:u_i\in A' \\ u_{i-2}\not\sim u_i, u_{i-1}\sim u_i}} |N(u_i,u_{i-1})|}_{(2)} + \underbrace{\sum_{\substack{i:u_i\in A' \\ u_{i-2}\sim u_i, u_{i-1}\not\sim u_i}} |N(u_i,u_{i-2})\setminus N(u_{i-1},u_{i-2})|}_{(3)} +$$

$$\underbrace{\sum_{\substack{i:u_i\in A' \\ u_{i-2}\sim u_i, u_{i-1}\sim u_i}} \Big(|N(u_i,u_{i-2}) \cup N(u_i,u_{i-1})|\Big)}_{(4)} +$$

$$\underbrace{\sum_{i:u_i\in A'} |N'(u_i)|}_{(1)} + \mathcal{O}(n^{2/3}\log\log n).$$

We claim that every vertex not lying on $P'$ is counted at most once in one of the sums (1) − (4). This will be proved by a case analysis.

(1) Let $x$ be some vertex which occurs in sum (1) for some $j$, i.e., $x \in N(u_j, u_{j-1})$. By definition of $N'$, $x$ is only adjacent to $u_j$ on $P$. Consequently, $x$ is only adjacent to one vertex on $P'$ and is counted once.

(2) Now let $x$ be some vertex which occurs in sum (2) for some $j$, i.e., $x \in N(u_j, u_{j-1})$. Thus, $|N(u_j, u_{j-2})| < n^{2/3}$ but $|N(u_j, u_{j-1})| \geq n^{2/3}$ and consequently $u_{j-1} \notin A'$. Hence, the only remaining possibility for $x$ to be counted in one of the four sums is as a common neighbor of $u_j$ and $u_{j+1}$. Since $u_j \in A'$, we have $u_j \not\sim u_{j+1}$. Hence, $x$ could be only counted in (3) with index $j+1$ further, but this is not possible as $x \notin N(u_{j+1}, u_{j-1})\setminus N(u_j, u_{j-1})$ due to $x \in N(u_j, u_{j-1})$.

(3) Assume that $x$ occurs in sum (3) for some $j$. Then we have $|N(u_j, u_{j-2})| \geq n^{2/3}$ but $|N(u_j, u_{j-1})| < n^{2/3}$. Hence, $x$ is adjacent to $u_{j-2}$ and $u_j$ on $P$ and could possibly be adjacent to $u_{j-1}$. However, these are the only possibilities, as otherwise, $P$ would not be a shortest path. The only remaining possibility for $x$ to be counted in one of the four sums is with a summation index $j-1$. However, it cannot be counted with a summation index $j-1$, as $x \notin N(u_{j-1}, u_{j-2})$ if $x$ is counted in sum (3).

(4) Finally, let $x$ be counted in sum (4) for some $j$. If $x$ is a common neighbor of $u_{j-2}, u_{j-1}$ and $u_j$, $x$ is only counted once, as $u_{j-1} \sim u_j$ and hence $u_{j-1} \notin A'$. Otherwise, $x$ could be a common neighbor of $u_{j-1}$, $u_j$ and $u_{j+1}$. Since $u_j \not\sim u_{j+1}$, $x$ could be only counted in sum (3) with a summation index $j+1$ further. However, as $x \notin N(u_{j+1}, u_{j-1})\setminus N(u_j, u_{j-1})$, $x$ is only counted once.

As before, let $X_{A'}$ be the number of rounds required for the information to reach from each node of $A'$ the corresponding successor on $P'$. By the previous argumentation we can estimate $X_{A'}$ as follows:

$$X_{A'} := \sum_{i:u_i \in A'} \mathsf{Geo}\left(\frac{1}{x_i}\right),$$

such that $\sum_{i:u_i\in A'} x_i = n + \tilde{\mathcal{O}}(n^{2/3})$. Therefore Lemma 3 yields for $n \ln n + n \ln n)^{2/3} \le y \le 12 n \log n$ that $\mathbf{Pr}\left[\, X_{A'} = n \ln n + n(\ln n)^{2/3} \,\right]$ is less than

$$\begin{aligned}
&(y + |A'| - 1)^{|A'|-1} \cdot e^{-\frac{y-|A'|}{n+\tilde{\mathcal{O}}(n^{2/3})}} (\gamma n)^{-|A'|} \\
&\le \left(\mathcal{O}(1) n \ln n\right)^{|A'|-1} \cdot \exp\left(-\frac{n \cdot (\ln n + \ln^{2/3} n)}{n + \tilde{\mathcal{O}}(n^{2/3})} + \frac{\mathcal{O}(\log\log n)}{n + \tilde{\mathcal{O}}(n^{2/3})}\right) \\
&\quad \cdot (\ln \ln n)^{\mathcal{O}(\log\log n)} \cdot n^{-|A'|} \\
&\le \tilde{\mathcal{O}}(1) \cdot n^{|A'|-1} \cdot (\ln n)^{\mathcal{O}(\log\log n)} \cdot \\
&\quad \exp\left(-(\ln n + \ln^{2/3} n) + \frac{\tilde{\mathcal{O}}(n^{-1/3}) \cdot (\ln n + \ln^{2/3} n)}{1 + \tilde{\mathcal{O}}(n^{-1/3})}\right) \cdot \\
&\quad (\ln \ln n)^{\mathcal{O}(\log\log n)} \cdot n^{-|A'|} \\
&= \tilde{\mathcal{O}}(1) \cdot n^{|A'|-1} \cdot n^{-|A'|} \cdot e^{-\ln n} \cdot e^{-\ln^{2/3} n} \cdot (\ln n)^{\mathcal{O}(\log\log n)} \cdot (\ln\ln n)^{\mathcal{O}(\log\log n)} \\
&= \tilde{\mathcal{O}}(1) \cdot n^{-2} \cdot e^{-\ln^{2/3} n} \cdot e^{\mathcal{O}(\log\log n)^2} \\
&= \tilde{\mathcal{O}}(n^{-2} \cdot e^{-\ln^{1/2} n}).
\end{aligned}$$

By the Union Bound, we have

$$\mathbf{Pr}\left[\, n \ln n + n(\ln n)^{2/3} \le X_{A'} \le 4n \ln n + 2n(\ln n)^{2/3} \,\right] \le \tilde{\mathcal{O}}(n^{-1} e^{-\ln^{1/2} n}).$$

Since $\mathbf{E}\left[\, X_{A'} \,\right] = \mathcal{O}(n)$ and $X_{A'}$ is the sum of geometrically distributed random variables, we have for a proper constant $\beta$ (cf. [9])

$$\mathbf{Pr}\left[\, X_{A'} \ge \beta n \ln n \,\right] \le \frac{1}{n^2}.$$

Therefore we conclude that

$$\mathbf{Pr}\left[\, X_{A'} \ge n \ln n + n(\ln n)^{2/3} \,\right] = \tilde{\mathcal{O}}(n^{-1} e^{-\ln^{1/2} n})$$

and together with the other parts of this proof we obtain for $X := X_{A'} + X_{A''} + X_B$

$$\begin{aligned}
&\mathbf{Pr}\left[\, X \ge n \ln n + n(\ln n)^{2/3} + 2n \ln\ln n + \frac{4}{\log\log n} \cdot n \ln n + n \,\right] \\
&= \tilde{\mathcal{O}}(n^{-1} e^{-\ln^{1/2} n}).
\end{aligned}$$

Consequently, an arbitrary node in $G$ receives the information after $n \ln n + n(\ln n)^{2/3} + \frac{4}{\log\log n} \cdot n \ln n$ rounds with probability $1 - \tilde{\mathcal{O}}(n^{-1} e^{-\ln^{1/2} n})$. Finally, using Markov's inequality gives

$$\begin{aligned}
&\mathbf{Pr}\left[H\Big(n\ln n + n(\ln n)^{2/3} + \frac{4}{\log\log n}\cdot n\ln n + n\Big) \geq 1\right] \\
&\leq \mathbf{E}\left[H\Big(n\ln n + n(\ln n)^{2/3} + \frac{4}{\log\log n}\cdot n\ln n + n\Big)\right] \\
&\leq \tilde{\mathcal{O}}(e^{-\ln^{1/2} n}),
\end{aligned}$$

and our claim follows. □

It is easy to see that the runtime in $K_{1,n-1}$ reduces to the Coupon-Collector-Problem [19] and it is well-known that $(1-o(1))n\ln n$ rounds are necessary with probability $1-o(1)$. Therefore, the above upper bound is tight.

### *4.2 Price of Randomness*

In this subsection, we compare the runtime of a fastest deterministic broadcasting algorithm with the runtime of the randomized broadcasting algorithm. Let $\mathsf{PR}(\mathcal{G}, n) = \max_{G\in\mathcal{G},|V(G)|=n} \frac{\mathsf{RT}(G,1/2)}{\mathsf{DT}(G)}$ for some graph class $\mathcal{G}$, where $\mathsf{DT}(G)$ is the runtime of a fastest deterministic algorithm in $G$.

**Theorem 5** *Let $\mathcal{R}$ be the set of regular graphs, and $\mathcal{G}$ the set of general graphs according to Section 2. Then, we have*

$$\mathsf{PR}(\mathcal{G}) = \Theta(n), \ \textit{and}\ \mathsf{PR}(\mathcal{R}) = \Theta\left(\frac{n}{\log n}\right).$$

*Proof.* Since $\mathsf{DT}(G) \geq \log_2 n$ and $\mathsf{RT}(G, \frac{1}{2}) \leq (1+o(1))n\log n$ by Theorem 4, it holds that $\mathsf{PR}(\mathcal{G}) = \mathcal{O}(n)$. To see that $\mathsf{PR}(\mathcal{G}) = \Omega(n)$ consider a complete graph $K_{n/2}$ in which every node is connected to an additional isolated vertex and let $G$ be this graph (of size $n$). It is easy to see that $\mathsf{DT}(G) \leq \lceil \log_2 n\rceil + 1$. Then let us consider the performance of $\mathcal{RBA}$. We may assume for simplicity that all vertices of the complete graph are informed and $\frac{n}{2}-1$ vertices of degree 1 are still uninformed. The probability that after further $n/2\ln(n/2) = \Theta(n\log n)$ steps all vertices have been informed equals

$$\left(1-\left(1-\frac{1}{n/2}\right)^{(n/2)\ln(n/2)}\right)^{n/2-1} \leq \left(1-\frac{2}{n}\right)^{n/2-1} \leq \frac{1}{2},$$

if $n \geq 4$. Let us now consider the second statement of the theorem. Proposition 1 implies that with constant probability every regular graph becomes completely informed within $\mathcal{O}(n)$ rounds. For the lower bound, consider two complete graphs $K_{n/2}$ and $K'_{n/2}$ with $n/2$ nodes. Connect some node $u \in K_{n/2}$ with a node $v \in K'_{n/2}$, and a node $u' \in K_{n/2},\ u' \neq u$ with a node $v' \in K'_{n/2},\ v' \neq v$. If we delete the edges $\{u, u'\}$ and $\{v, v'\}$, then the resulting graph is regular. It is easy to see that a deterministic algorithm broadcasts any information in this graph in time $\Theta(\log n)$, however, the push algorithm needs at least $\Omega(n)$ rounds, since the edge between $K_{n/2}$ and $K'_{n/2}$ is chosen in some step only with probability $\mathcal{O}(1/n)$. □

## 5 Robustness of Randomized Broadcasting and Applications

In this section we analyze the robustness of the push algorithm against random node failures. Then, we use the results of this analysis to determine the runtime of agent based broadcasting in graphs with good local expansion properties and to derive new bounds on the runtime of randomized broadcasting in Cartesian products of graphs.

### *5.1 Robustness*

In this section we consider the robustness of the push algorithm against random failures. We assume here that in each round $t$, any informed node is allowed to fail with probability $1 - p$ for some $p \in (0, 1)$, independently of any failure in other rounds. However, there might exist failure dependencies between nodes within one round. We should note that our model is somehow a generalization of the probabilistic failure-model examined in [15], in which no dependencies between failures within the same round are allowed.

As described above, only informed vertices are allowed to fail. If an informed vertex fails in some round $t$, then it does not choose any communication partner to send the message. If it is functional, then it executes the push algorithm as described in the introduction. If some informed node is able to send a message, then we assume that the transmission will be completed.

We should note here that the results below can be simply extended to the case when restricted dependencies are allowed between the time steps (e.g. if a node fails in some step $t + 1$ after being functional in step $t$, then it fails for $\mathcal{O}(1)$ further rounds). Therefore, this model is well-suited to describe restricted asynchronicity in a network, in which even if some nodes are busy for a time period, the messages sent to these nodes do not get lost.

Denote by $\mathsf{RT}'(G, p') = \min\{t \in I\!N \mid \mathbf{Pr}\,[\,I(t) = V\,] \geq p'\}$, the runtime of the push-algorithm in the previously described failure model. Now we can state the following theorem.

**Theorem 6** *For any graph $G$ it holds that*

$$\mathsf{RT}'(G, 1-\mathcal{O}(n^{-1})) \leq \frac{6}{p} \cdot \mathsf{RT}(G, 1-\mathcal{O}(n^{-1})).$$

*Proof.* In this proof, we are going to show that any instance of the push algorithm in the failure model can be related to an instance of the push algorithm without failures. Then, we show that, with very high probability, there is no large difference between the runtimes of the corresponding instances.

For an instance $T$ of the push algorithm (in the model without failures) let $N_{T,j}(v)$ denote the neighbor of $v$ chosen in step $i(v)+j$, where $i(v)$ denotes the time step in which $v$ has been informed (according to instance $T$). Accordingly, let $(N_{T,j}(v))_{j=1}^{\infty}$ be the sequence of nodes chosen by $v$ in steps $i(v)+1,\ldots,\infty$. Similarly, for any instance $T'$ of the push algorithm in the failure model, let $N'_{T',j}(v)$ denote the neighbor of $v$ chosen in step $i'(v)+X_{T',j}(v)+j$, where $i'(v)$ denotes the time step in which $v$ has been informed according to $T'$ and $X_{T',j}(v)$ is the number of failures of $v$ before $v$ has been functional $j$ times, i.e., the number of failures within the first $X_{T',j}(v)+j$ steps after $v$ has become informed. Again, let $(N'_{T',j}(v))_{j=1}^{\infty}$ be the sequence of nodes chosen by $v$ in the steps $v$ is functional. Furthermore, let $\mathsf{RT}(T)$ be the *exact* runtime of the push algorithm for instance $T$, and let $\mathsf{RT}'(T')$ be the exact runtime of the push algorithm (in the failure model) for instance $T'$. Hereby, an instance $T$ of the push algorithm is completely described by the set of sequences $\cup_{v\in V}(N_{T,j}(v))_{j=1}^{\infty}$ and the node informed at the beginning. However, an instance $T'$ is only described by both sets of sequences $\cup_{v\in V}(N'_{T',j}(v))_{j=1}^{\infty}$, $\cup_{v\in V}(X_{T',j}(v))_{j=1}^{\infty}$, and the node informed at the beginning. In the following paragraphs, $N_{T,j}(v)$ is simply denoted by $N_j(v)$ for any $j$ and $v$, and we write $\mathsf{RT}(\cup_{v\in V}(N_j(v))_{j=1}^{\infty})$ instead of $\mathsf{RT}(T)$. Let now $\mathcal{T}'(\cup_{v\in V}(N_j(v))_{j=1}^{\infty})$ denote the set of instances in the failure model, which contain the set of sequences $\cup_{v\in V}(N_j(v))_{j=1}^{\infty}$. Now we are going to show for any set of sequences $\cup_{v\in V}(N_j(v))_{j=1}^{\infty}$ that an instance $T'\in\mathcal{T}'(\cup_{v\in V}(N_j(v))_{j=1}^{\infty})$, which has a running time of at most $6/p\cdot\mathsf{RT}(\cup_{v\in V}(N_j(v))_{j=1}^{\infty})$ occurs with probability at least $1-n^{-1}$ in $\mathcal{T}'(\cup_{v\in V}(N_j(v))_{j=1}^{\infty})$.

To show this, we first consider the push algorithm without failures, and analyze for the instance $\cup_{v\in V}(N_j(v))_{j=1}^{\infty}$ the path used by the information to reach from the initially informed node $s$ a node $v$. Let $P(s,v) := (s=u_1,u_2,\ldots,u_l=v)$ be this path, and define $d_j := i(u_{j+1})-i(u_j)$ as the time the information needs to be sent from $u_j$ to $u_{j+1}$. Let $d(P(s,v)) := \sum_{j=1}^{l-1} d_j = i(v)$. Since $\log_2 n$ is a lower bound on the runtime of the push algorithm (e.g. [9]), we have $\max_{v\in V} d(P(u,v)) = \mathsf{RT}(\cup_{v\in V}(N_j(v))_{j=1}^{\infty}) \geq \log_2 n$.

Now we consider some instance $T'$ containing $\cup_{v\in V}(N_j(v))_{i=1}^{\infty}$. Obviously, the path $P(s,v) := (s = u_1, u_2, \ldots, u_l = v)$ still exists in $T'$, however, the time needed for the information to reach $u_{i+1}$ from $u_i$ is given by $d'_j := i'(u_{j+1}) - i'(u_j) = X_{T',d_j}(u_j) + d_j$. Let $d'(P(s,v)) := \sum_{j=1}^{l-1}(d_j + X_{T',d_j}(u_j))$, then $\mathsf{RT}'(T') \leq \max_{v\in V} d'(P(s,v))$ (there might exist faster paths, but it is sufficient here to focus only on $P(s,v)$). In order to estimate $d'(P(s,v))$ we define for any time step $t$ the random variable $X_t$, which is 0 if the last node being informed at time $t-1$ on $P(s,v)$ fails in step $t$, and 1 otherwise. Since all nodes fail in some round with probability $1-p$, independently of the other rounds, $X_t = 0$ with probability $1-p$ and $X_t = 1$ with probability $p$, independently of $X_j$ for any $j \neq t$.

For the next, we show that $\sum_{t=1}^{6/p\cdot\max\{d(P(s,v)),\log_2 n\}} X_t \geq d(P(s,v))$ with probability at least $1 - n^{-2}$. Since all $X_t$ are independent from each other, we use the Chernoff bounds [4, 12] with $\delta = 5/6$, and obtain

$$\mathbf{Pr}\left[\sum_{t=1}^{6/p\cdot d(P(s,v))} X_t \leq \left(1 - \frac{5}{6}\right) 6d(P(s,v))\right] \leq e^{\frac{-6d(P(s,v))25}{72}} \leq e^{-2d(P(s,v))} \leq n^{-2},$$

whenever $d(P(s,v)) \geq \log_2 n$. This implies that if $d(P(s,v)) \geq \log_2 n$, then $\sum_{j=1}^{k-1}(d_j + X_{T',d_j}(u_j)) \leq 6/p \cdot d(P(s,v))$ with probability $1 - \mathcal{O}(n^{-2})$. If now $d(P(s,v)) < \log_2 n$, then by replacing $d(P(u,v))$ with $\log_2 n$ in the inequality above we obtain

$$\mathbf{Pr}\left[\sum_{t=1}^{6/p\cdot\log_2 n} X_t \leq \log_2 n\right] \leq n^{-2}.$$

This implies that $d'(P(s,v)) \leq 6/p \cdot \max\{d(P(s,v)), \log_2 n\}$ with probability $1 - \mathcal{O}(n^{-2})$ for a node $v \in V$. Using the Union bound we obtain the result for any node $v \in V$ with probability $1 - \mathcal{O}(n^{-1})$. Since $\max_{v\in V} d(P(s,v)) = \mathsf{RT}(\cup_{v\in V}(N_j(v))_{j=1}^{\infty}) \geq \log_2 n$, the claim follows.

To conclude the proof, we use the fact that any instance $\cup_{v\in V}(N_j(v))_{j=1}^{\infty}$ occurs with the same probability in the model without failures, and for some fixed $\cup_{v\in V}(X_{T',j}(v))_{j=1}^{\infty}$ two instances $\cup_{v\in V}(N_j(v))_{j=1}^{\infty}$, $\cup_{v\in V}(X_{T',j}(v))_{j=1}^{\infty}$ and $\cup_{v\in V}(N'_j(v))_{j=1}^{\infty}$, $\cup_{v\in V}(X_{T',j}(v))_{j=1}^{\infty}$ occur with the same probability in the failure model, too. Thus,

$$\mathsf{RT}'(G, 1 - \mathcal{O}(n^{-1})) \leq \frac{6}{p} \cdot \mathsf{RT}(G, 1 - \mathcal{O}(n^{-1})).$$

□

In the following paragraphs, we use Theorem 6 to derive bounds on the runtime of agent based broadcasting [6] (cf. Section 1 for the description of the algorithm), which is denoted in the following paragraphs by $\mathsf{AT}(G, 1-\mathcal{O}(n^{-1}))$. The parameters $G$ and $1-\mathcal{O}(n^{-1})$ have the same role as in $\mathsf{RT}(G, 1-\mathcal{O}(n^{-1}))$. We assume that in the beginning round all agents are distributed independently according to the stationary distribution. However, the distribution of the agents in some round $t+1$ depends on their distribution in round $t$. In order to derive a strong relationship between $\mathsf{AT}(G, 1-\mathcal{O}(n^{-1}))$ and $\mathsf{RT}(G, 1-\mathcal{O}(n^{-1}))$, we introduce so-called log-expanding graphs. Their good expansion properties allow us to cope with these dependencies.

**Definition 2** *We call a graph $G$, where $\Delta = O(\delta)$, a* log*-expanding graph, if*

$$\forall v \in V(G)\ \exists \gamma = \mathcal{O}(1) : |B_\gamma(v)| \geq \max\{(c \cdot \delta(G))^\gamma, 5 \ln n\}$$

*where* $B_\gamma(v) := \{w \in V(G) \mid \operatorname{dist}(w, v) = \gamma\}$.

Among others, Random-Graphs [2], Hypercubes [13] and Star graphs [7] are log-expanding graphs. In [9] and [7] it has been shown that on all these graphs $\mathcal{RBA}$ has an optimal runtime of $\mathcal{O}(\log n)$.

**Theorem 7** *Let $G$ be a log-expanding graph and assume that $n$ agents are initially distributed independently and according to the stationary distribution. Then,*

$$\mathsf{AT}(G, 1-\mathcal{O}(n^{-1})) \leq \mathcal{O}(\mathsf{RT}(G, 1-\mathcal{O}(n^{-1}))).$$

*Proof.* Note, that the stationary distribution of a random walk is given by $\pi(v) = \frac{\deg(v)}{2|E|}$ for any node $v \in V(G)$. Hence, for any time step $t$ an agent is lying on some vertex $v$ with probability $\pi(v)$. Now consider some node $v$ together with $B_\gamma(v)$. Then, the probability for some agent to be located on $B_\gamma(v)$ at step $t$ is

$$\pi(B_\gamma(v)) = \frac{\sum_{u \in B_\gamma(v)} \deg(u)}{2|E|} \geq \frac{|B_\gamma(v)|\delta}{n\Delta} = \Omega\left(\frac{\delta(G)^\gamma}{n}\right).$$

Since we have $n$ agents, the expected number of agents located in $B_\gamma(v)$ at some time $t$ equals $\Omega(\delta(G)^\gamma)$. Due to $|B_\gamma(v)| \geq \max\{(c \cdot \delta(G))^\gamma, 5 \ln n\}$, we may apply a Chernoff bound to conclude that there are $\Omega(\delta(G)^\gamma)$ agents in $B_\gamma(v)$ with probability $1-\mathcal{O}(n^{-5})$. This guarantees that in each time step within some first $\mathcal{O}(n^2)$ rounds, there are $\Omega(\delta(G)^\gamma)$ agents in $B_\gamma(v)$, with probability $1-\mathcal{O}(n^{-3})$. Observe that some fixed agent in $B_\gamma(v)$ reaches the vertex $v$ within $\gamma$ steps with probability $\Delta(G)^{-\gamma}$ and recall that all agents are performing independent random walks. Thus, given that for any $t \leq c'\mathsf{RT}(G, 1-\mathcal{O}(n^{-1}))$, with $c'$ being an arbitrary constant, there are $(c \cdot \delta(G))^\gamma$ agents in $B_\gamma(v)$ at time $t$, node $v$ will be visited by at least one agent with constant probability within the time interval $[t, t+\gamma]$.

In order to relate $\mathsf{AT}(G, 1-\mathcal{O}(n^{-1}))$ to $\mathsf{RT}(G, 1-\mathcal{O}(n^{-1}))$ let $\cup_{v\in V}(N_j(v))_{j=1}^\infty$ define an instance of the original push algorithm and let $\cup_{v\in V}(N_j(v))_{j=1}^\infty$, $\cup_{v\in V}(N'_j(v))_{j=1}^\infty$ define an instance of the agent based algorithm. Here, $N_j(v)$ denotes the neighbor of $v$ chosen by the $j$th agent which visits $v$ *after $v$ has become informed.* $N'_j(v)$ denotes the neighbor of $v$ chosen by the $j$th agent which visits $v$ *before $v$ becomes informed.* We should note here that if several agents visit the same node in the same time step, then we may choose some random ordering for these agents.

In order to show the theorem, we simply apply the fact that if for any $t = \mathcal{O}(n^2)$ there are $\Theta(|B_\gamma(v)|)$ agents in $B_\gamma(v)$ at time $t$, then any node is visited in some time interval $[t, t+\mathcal{O}(1)]$ with at least some constant probability $p$, regardless of the placement of the agents in some step before $t$. Hence, a node $v$ forwards the information with constant probability, in any time interval $[t, t+\mathcal{O}(1)]$, according to an instance $(N_j(v))_{j=1}^\infty$. Since each instance $\cup_{v\in V}(N_j(v))_{j=1}^\infty$ occurs with the same probability in the push model, and for a fixed $\cup_{v\in V}(N'_j(v))_{j=1}^\infty$ two instances $\cup_{v\in V}(N^1_j(v))_{j=1}^\infty$, $\cup_{v\in V}(N'_j(v))_{j=1}^\infty$ and $\cup_{v\in V}(N^2_j(v))_{j=1}^\infty$, $\cup_{v\in V}(N'_j(v))_{j=1}^\infty$ occur with the same probability in the agent based model, using the same arguments as in Theorem 6 we obtain the theorem. $\square$

Using Theorem 6 we also state new results on the runtime of randomized broadcasting in Cartesian products of graphs [11]. We denote by $G_1 \times G_2$ the product of two connected graphs, $G_1$ and $G_2$ of size $n_1 := |V(G_1)|$ and $n_2 := |V(G_2)|$, resp. It is easy to see that $\mathsf{DT}(G_1 \times G_2) \le \mathsf{DT}(G_1) + \mathsf{DT}(G_2)$. For the randomized case, we can state the following.

**Theorem 8** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two arbitrary graphs, where $n_1 \ge 2$, and $n_2 \ge 2$. For $p := \min\{\frac{\delta_1}{\delta_1+\Delta_2}, \frac{\delta_2}{\Delta_1+\delta_2}\}$ it holds that*

$$\mathsf{RT}(G_1 \times G_2, 1-n^{-1}) \le \frac{1}{p} \cdot \mathcal{O}\left(\mathsf{RT}(G_1, 1-n_1^{-1}) + \mathsf{RT}(G_2, 1-n_2^{-1})\right).$$

*Proof.* Without loss of generality, we assume that $|n_1| \ge \sqrt{n}$. Let $(u_0, v_0)$ be the initially informed node. Applying Theorem 6, $G_1 \times \{v_0\}$ becomes completely informed after $\frac{6}{p} \cdot \mathsf{RT}(G_1, 1-n_1^{-1})$ steps, with probability $1-\mathcal{O}(n_1^{-1}) = 1-\mathcal{O}(n^{-1/2})$. Thus, after $\frac{25}{p} \cdot \mathsf{RT}(G_1, 1-n_1^{-1})$ rounds $G_1 \times \{v_0\}$ is completely informed, with probability $1-n^{-2}$. Now we consider all informed nodes of $V_1 \times \{v_0\}$ simultaneously. Again, we can use Theorem 6 to conclude that after

$\frac{25}{p} \cdot \mathsf{RT}(G_2, 1-n_2^{-1})$ rounds any graph $\{u\} \times V_2, u \in V_1$ is completely informed, with probability $1-(n_2^{-1})^4 \geq 1-(\frac{1}{2})^4 = 1-\frac{1}{16}$. Hence at least one subgraph $\{u'\} \times V_2$, for some fixed $u' \in V_1$, is completely informed with probability $1-(\frac{1}{16})^{\sqrt{n}} \geq 1-n^{-1}$. Hence, for any $v \in V_2$, the subgraph $G_1 \times \{v\}$ contains at least one informed node $(u(v), v)$. Now we can use the same arguments as above to conclude that one of these nodes $(u(v), v)$ informs the whole subgraph $G_1 \times \{v\}$ within $\frac{50}{p} \cdot \mathsf{RT}(G_1, 1-n_1^{-1})$ rounds with probability $1-n^{-3}$. Here, we ignore any transmissions along edges induced by $G_2$ and hence we may assume independence between the subgraphs $G_1 \times \{v\}$ for any $v$. Since we have at most $\sqrt{n}$ of these subgraphs, the information is completely distributed among all these subgraphs with probability $1-n^{-2}$. Now, if $n \geq 4$, then it holds that $(1-n^{-2})^3 \geq 1-3n^{-2} \geq 1-n^{-1}$, and the claim follows. □

## 6 Conclusion

In this paper, we derived several tight lower and upper bounds for the runtime of the push algorithm. First, we obtained a lower bound of $\log_{2-\frac{1}{\Delta}} n + \log_{(\frac{\Delta}{\Delta-1})^\Delta} n - o(\log\log n) \approx 1.69 \log_2 n$ for regular graphs. Together with a previous result of Pittel [20] this bound implies that the push algorithm has fastest performance on complete graphs, if we neglect the small error term of $o(\log\log n)$. For non-regular graphs we established a lower bound of $\log_{2-\frac{1}{\Delta}} n + \log_4 n - o(\log n) \approx 1.5 \log_2 n$. An open problem here is to close the gap between these two lower bounds or to find a non-regular graph on which the randomized broadcasting algorithm performs substantially better than on complete graphs. In the second part of this paper we proved a tight upper bound of $(1+o(1))n \ln n$ which significantly improves a previous result of [9]. In the second part of this paper, we have shown that the push algorithm is robust against random failures, and analyzed the performance of agent based broadcasting in so called log-expanding graphs. Finally, we also derived tight bounds for the runtime of the push algorithm in Cartesian products of graphs.

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