

# A Statistical Perspective on Randomized Sketching for Ordinary Least-Squares

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## Abstract

We consider statistical aspects of solving large-scale least-squares (LS) problems using randomized sketching algorithms. For a LS problem with input data  $(X, Y) \in \mathbb{R}^{n \times p} \times \mathbb{R}^n$ , where  $n$  and  $p$  are both large and  $n \gg p$ , sketching algorithms use a “sketching matrix,”  $S \in \mathbb{R}^{r \times n}$ , where  $r \ll n$ , e.g., a matrix representing the process of random sampling or random projection. Then, rather than solving the LS problem using the full data  $(X, Y)$ , sketching algorithms solve the LS problem using only the “sketched data”  $(SX, SY) \in \mathbb{R}^{r \times p} \times \mathbb{R}^r$ . Prior work has typically adopted an *algorithmic perspective*, in that it has made no statistical assumptions on the input  $X$  and  $Y$ , and instead it has assumed that the data  $(X, Y)$  are fixed and worst-case. In this paper, we adopt a *statistical perspective*, and we consider the mean-squared error performance of randomized sketching algorithms, when data  $(X, Y)$  are generated according to a statistical linear model  $Y = X\beta + \epsilon$ , where  $\epsilon$  is a noise process. To do this, we first develop a framework for assessing, in a unified manner, algorithmic and statistical aspects of randomized sketching methods. We then consider the statistical prediction efficiency (SPE) and the statistical residual efficiency (SRE) of the sketched LS estimator; and we use our framework to provide results for several types of random projection and random sampling sketching algorithms. Among other results, we show that the SRE can be bounded when  $p \lesssim r \ll n$  but that the SPE typically requires the sample size  $r$  to be substantially larger. Our theoretical results reveal that, depending on the specifics of the situation, leverage-based sampling methods can perform as well as or better than projection methods. Our empirical results reveal that when  $r$  is only slightly greater than  $p$  and much less than  $n$ , projection-based methods out-perform sampling-based methods, but as  $r$  grows, sampling methods start to out-perform projection methods.

## 1 Introduction

Recent work in large-scale data analysis has focused on developing so-called sketching algorithms: given a data set and an objective function of interest, construct a small “sketch” of the full data set, e.g., by using random sampling or random projection methods, and use that sketch as a surrogate to perform computations of interest for the full data set (see [20] for a review). Most effort in this area has adopted an *algorithmic perspective*, whereby one shows that, when the sketches are constructed appropriately, one can obtain answers that are approximately as good as the exact answer for the input data at hand, in less time than would be required to compute an exact answer for the data at hand. In statistics, however, one is often more interested in how well a procedure performs relative to an hypothesized model than how well it performs on the particular data set at hand. Thus, in this paper, we adopt a *statistical perspective* on recently-developed randomized sketching algorithms, and we provide bounds on two related statistical objectives for several types of random projection and random sampling sketching algorithms.

## 1.1 Overview of the problem

The problem we consider in this paper is the ordinary least-squares (LS or OLS) problem: given as input a matrix  $X \in \mathbb{R}^{n \times p}$  of observed features or covariates and a vector  $Y \in \mathbb{R}^n$  of observed responses, return as output a vector  $\beta_{OLS}$  that solves the following optimization problem:

$$\beta_{OLS} = \arg \min_{\beta \in \mathbb{R}^p} \|Y - X\beta\|_2^2. \quad (1)$$

We will assume that  $n$  and  $p$  are both very large, with  $n \gg p$ , and for simplicity we will assume  $\text{rank}(X) = p$ , e.g., to ensure a unique full-dimensional solution. The LS solution,  $\beta_{OLS} = (X^T X)^{-1} X^T Y$ , has a number of well-known desirable statistical properties [4]; and it is also well-known that the running time or computational complexity for this problem is  $O(np^2)$  [10].<sup>1</sup> For many modern applications, however,  $n$  may be on the order of  $10^6 - 10^9$  and  $p$  may be on the order of  $10^3 - 10^4$ , and thus computing the exact LS solution with traditional  $O(np^2)$  methods can be computationally challenging. This, coupled with the observation that approximate answers often suffice for downstream applications, has led to a large body of work on developing fast approximation algorithms to the LS problem [20].

One very popular approach to reducing computation is to perform LS on a carefully-constructed “sketch” of the full data set. That is, rather than computing a LS estimator from Problem (1) from the full data  $(X, Y)$ , generate “sketched data”  $(SX, SY)$  where  $S \in \mathbb{R}^{r \times n}$ , with  $r \ll n$ , is a “sketching matrix,” and then compute a LS estimator from the following sketched problem:

$$\beta_S \in \arg \min_{\beta \in \mathbb{R}^p} \|SY - SX\beta\|_2^2. \quad (2)$$

Once the sketching operation has been performed, the additional computational complexity to compute the LS estimator with Problem (2) is  $O(rp^2)$ , i.e., simply call a traditional LS solver on the sketched problem. Thus, when using a sketching algorithm, two criteria are important: first, ensure the accuracy of the sketched LS estimator is comparable to, e.g., not much worse, than the performance of the original LS estimator; and second, ensure that computing and applying the sketching matrix  $S$  is not too computationally intensive, e.g., that is faster than solving the original problem exactly.

## 1.2 Prior approaches

Random sampling and random projections provide two approaches to construct sketching matrices  $S$  that satisfy both of these criteria and that have received attention recently in the computer science community. Very loosely speaking, a random projection matrix  $S$  is a dense matrix,<sup>2</sup> where each entry is a mean-zero bounded-variance Gaussian or Rademacher random variable, although other constructions based on randomized Hadamard transformations are also of interest; and a random sampling matrix  $S$  is a very sparse matrix that has exactly 1 non-zero entry (which typically equals one multiplied by a rescaling factor) in each row, where that one non-zero can be chosen uniformly, nonuniformly based on hypotheses about the data, or nonuniformly based on empirical statistics of the data such as the leverage scores of the matrix  $X$ . In particular, note that a sketch constructed from an  $r \times n$  random projection matrix  $S$  consists of  $r$  linear combinations of most or all of the rows of  $(X, Y)$ , and a sketch constructed from a random sampling matrix  $S$  consists of  $r$  typically-rescaled rows of  $(X, Y)$ . Random projection algorithms have received a great deal

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<sup>1</sup>That is,  $O(np^2)$  time suffices to compute the LS solution from Problem (1) for arbitrary or worst-case input, with, e.g., the Cholesky Decomposition on the normal equations, with a QR decomposition, or with the Singular Value Decomposition [10].

<sup>2</sup>The reader should, however, be aware of recently-developed input-sparsity time random projection methods [5, 22, 25].

of attention more generally, largely due to their connections with the Johnson-Lindenstrauss lemma [16] and its extensions; and random sampling algorithms have received a great deal of attention, largely due to their applications in large-scale data analysis applications [21]. A detailed overview of random projection and random sampling algorithms for matrix problems may be found in the recent monograph [20]. Here, we briefly summarize the most relevant aspects of the theory.

In terms of running time guarantees, the running time bottleneck for random projection algorithms for the LS problem is the application of the projection to the input data, i.e., actually performing the matrix-matrix multiplication to implement the projection and compute the sketch. By using fast Hadamard-based random projections, however, Drineas et al. [8] developed a random projection algorithm that runs on arbitrary or worst-case input in  $o(np^2)$  time. (See [8] for a precise statement of the running time, including the dependence on the error parameter  $\kappa$  below.) As for random sampling, it is trivial to implement uniform random sampling, but it is very easy to show examples of input data on which uniform sampling performs very poorly. On the other hand, Drineas et al. [9, 7] have shown that if the random sampling is performed with respect to nonuniform importance sampling probabilities that depend on the *empirical statistical leverage scores* of the input matrix  $X$ , i.e., the diagonal entries of the *hat matrix*  $H = X(X^T X)^{-1} X^T$ , then one obtains a random sampling algorithm that achieves much better results for arbitrary or worst-case input.

Leverage scores have a long history in robust statistics and experimental design. In the robust statistics community, samples with high leverage scores are typically flagged as potential outliers (see, e.g., [3, 4, 11, 13, 14]). In the experimental design community, samples with high leverage have been shown to improve overall efficiency, provided that the underlying statistical model is accurate (see, e.g., [26, 28]). This should be contrasted with their use in theoretical computer science. From the algorithmic perspective of worst-case analysis, that was adopted by Drineas et al. [8] and Drineas et al. [7], samples with high leverage tend to contain the most important information for subsampling/sketching, and thus it is beneficial for worst-case analysis to bias the random sample to include samples with large statistical leverage scores or to rotate to a random basis where the leverage scores are approximately uniformized.

The running-time bottleneck for this leverage-based random sampling algorithm is the computation of the leverage scores of the input data; and the obvious well-known algorithm for this involves  $O(np^2)$  time to perform a QR decomposition to compute an orthogonal basis for  $X$  [10]. By using fast Hadamard-based random projections, however, Drineas et al. [7] showed that one can compute approximate QR decompositions and thus approximate leverage scores in  $o(np^2)$  time, and (based on previous work [9]) this immediately implies a leverage-based random sampling algorithm that runs on arbitrary or worst-case input in  $o(np^2)$  time [7]. (See [7] for a precise statement of the running time, including the dependence on the error parameter  $\kappa$  below.) Readers interested in the practical performance of these randomized algorithms should consult BENDENPIK [1] or LSRN [23].

In terms of accuracy guarantees, both Drineas et al. [8] and Drineas et al. [7] prove that their respective random projection and leverage-based random sampling LS sketching algorithms each achieve the following worst-case error (WCE) guarantee: for any arbitrary  $(X, Y)$ ,

$$\|Y - X\beta_S\|_2^2 \leq (1 + \kappa)\|Y - X\beta_{OLS}\|_2^2, \quad (3)$$

with high probability for some pre-specified error parameter  $\kappa \in (0, 1)$ .<sup>3</sup> This  $1 + \kappa$  relative-error guarantee<sup>4</sup> is extremely strong, and it is applicable to arbitrary or worst-case input. That is, whereas in statistics one typically assumes a model, e.g., a standard linear model on  $Y$ ,

$$Y = X\beta + \epsilon, \quad (4)$$

<sup>3</sup>The quantity  $\|\beta_S - \beta_{OLS}\|_2^2$  is also bounded by Drineas et al. [8] and Drineas et al. [7].

<sup>4</sup>The nonstandard parameter  $\kappa$  is used here for the error parameter since  $\epsilon$  is used below to refer to the noise or error process.

where  $\beta \in \mathbb{R}^p$  is the true parameter and  $\epsilon \in \mathbb{R}^n$  is a standardized noise vector, with  $\mathbb{E}[\epsilon] = 0$  and  $\mathbb{E}[\epsilon\epsilon^T] = I_{n \times n}$ , in Drineas et al. [8] and Drineas et al. [7] no statistical model is assumed on  $X$  and  $Y$ , and thus the running time and quality-of-approximation bounds apply to any arbitrary  $(X, Y)$  input data.

### 1.3 Our approach and our main results

In this paper, we adopt a statistical perspective on these randomized sketching algorithms, and we address the following fundamental questions. First, under a standard linear model, e.g., as given in Eqn. (4), what properties of a sketching matrix  $S$  are sufficient to ensure low statistical, e.g., mean-squared, error? Second, how do existing random projection algorithms and leverage-based random sampling algorithms perform by this statistical measure? Third, how does this relate to the properties of a sketching matrix  $S$  that are sufficient to ensure low worst-case error, e.g., of the form of Eqn. (3), as has been established previously [8, 7, 20]? We address these related questions in a number of steps.

In Section 2, we will present a framework for evaluating the algorithmic versus statistical properties of randomized sketching methods in a unified manner; and we will show that providing worst-case error bounds of the form of Eqn. (3) and providing bounds on two related statistical objectives boil down to controlling different structural properties of how the sketching matrix  $S$  interacts with the left singular subspace of the design matrix. In particular, we will consider the oblique projection matrix,  $\Pi_S^U = U(SU)^\dagger S$ , where  $(\cdot)^\dagger$  denotes the Moore-Penrose pseudo-inverse of a matrix and  $U$  is the left singular matrix of  $X$ . This framework will allow us to draw a comparison between the worst-case WCE and two related statistical efficiency criteria, the statistical prediction efficiency (SPE) (which is based on the prediction error  $\mathbb{E}[\|X(\hat{\beta} - \beta)\|_2^2]$  and which is given in Eqn. (7) below) and the statistical residual efficiency (SRE) (which is based on residual error  $\mathbb{E}[\|Y - X\hat{\beta}\|_2^2]$  and which is given in Eqn. (8) below); and it will allow us to provide sufficient conditions that any sketching matrix  $S$  must satisfy in order to achieve good performance on these two statistical objectives.

In Section 3, we will present our main theoretical results, which consist of bounds for these two statistical quantities for variants of random sampling and random projection sketching algorithms. In particular, we provide upper bounds on the SPE and SRE (as well as the worst-case WCE) for four sketching schemes: (1) a vanilla leverage-based random sampling algorithm, as is analyzed by Drineas et al. [7]; (2) a variant of leverage-based random sampling, where the random samples are *not* re-scaled prior to their inclusion in the sketch, as is considered by Ma et al. [18, 19]; (3) a vanilla random projection algorithm, where  $S$  is a random matrix containing i.i.d. Gaussian or Rademacher random variables, as is popular in statistics and scientific computing; and (4) a random projection algorithm, where  $S$  is a random Hadamard-based random projection, as analyzed in [2]. For sketching schemes (1), (3), and (4), our upper bounds for each of the two measures of statistical efficiency are identical up to constants; and they show that the SRE scales as  $1 + \frac{p}{r}$ , while the SPE scales as  $\frac{p}{r}$ . In particular, this means that it is possible to obtain good bounds for the SRE when  $p \lesssim r \ll n$  (in a manner similar to the sampling complexity of the WCE bounds); but in order to obtain even near-constant bounds for SPE,  $r$  must be at least of constant order compared to  $n$ . For the sketching scheme (2), we show, on the other hand, that under the strong assumption that there are  $k$  “large” leverage scores and the remaining  $n - k$  are “small,” then the WCE scales as  $1 + \frac{p}{r}$ , the SRE scales as  $1 + \frac{pk}{rn}$ , and the SPE scales as  $\frac{k}{r}$ . That is, sharper bounds are possible for leverage-score sampling without re-scaling in the statistical setting, but much stronger assumptions are needed on the input data.

In Section 4, we will supplement our theoretical results by presenting our main empirical results, which consist of an evaluation of the complementary properties of random sampling versus random projection methods. Among other things, our empirical results show that for  $r$  larger than  $p$  but much closer to  $p$  than  $n$ , projection-based methods tend to out-perform sampling-based methods, while for  $r$  significantly larger

than  $p$ , our leverage-based sampling methods perform slightly better. Furthermore, our empirical results also confirm that in the statistical setting, scheme (2) clearly out-performs scheme (1), when the strong assumptions of the former are satisfied; and for  $r$  much larger than  $p$ , it also out-performs projection-based approaches, which is also consistent with the theoretical results.

We will present proofs of our main results in Section 5; and in Section 6, we will provide a brief discussion and conclusion.

## 1.4 Additional related work

Very recently, Ma et al. [18, 19] considered statistical aspects of leverage-based sampling algorithms (called algorithmic leveraging in [18, 19]). Assuming a standard linear model on  $Y$  of the form of Eqn. (4), the authors developed first-order Taylor approximations to the statistical relative efficiency of different estimators computed with leverage-based sampling algorithms, and they verified the quality of those approximations with computations on real and synthetic data. Taken as a whole, their results suggest that, if one is interested in the statistical performance of these randomized sketching algorithms, then there are nontrivial trade-offs that are not taken into account by standard worst-case analysis. Their approach, however, does not immediately apply to random projections or other more general sketching matrices. Further, the realm of applicability of the first-order Taylor approximation was not precisely quantified, and they left open the question of structural characterizations of random sketching matrices that were sufficient to ensure good statistical properties on the sketched data. We address these issues in this paper.

## 2 General framework and structural results

In this section, we first develop a framework that allows us to view the algorithmic and statistical perspectives on LS problems from a common perspective. We then use this framework to show that existing worst-case bounds as well as our novel statistical bounds for the mean-squared errors can be expressed in terms of different structural conditions on how the sketching matrix  $S$  interacts with the data  $(X, Y)$ .

### 2.1 A statistical-algorithmic framework

Recall that we are given as input a data set,  $(X, Y) \in \mathbb{R}^{n \times p} \times \mathbb{R}^n$ , and the objective function of interest is the standard LS objective, as given in Eqn. (1). Since we are assuming, without loss of generality, that  $\text{rank}(X) = p$ , we have that

$$\beta_{OLS} = X^\dagger Y = (X^T X)^{-1} X^T Y, \tag{5}$$

where  $(\cdot)^\dagger$  denotes the Moore-Penrose pseudo-inverse of a matrix and the second equality follows since  $\text{rank}(X) = p$ .

To present our framework and objectives, let  $S \in \mathbb{R}^{r \times n}$  denote an *arbitrary* sketching matrix. That is, although we will be most interested in sketches constructed from random sampling or random projection operations, for now we let  $S$  be *any*  $r \times n$  matrix. Then, we are interested in analyzing the performance of objectives characterizing the quality of a “sketched” LS objective, as given in Eqn (2), where again we are interested in solutions of the form

$$\beta_S = (SX)^\dagger SY. \tag{6}$$

(We emphasize that this does *not* in general equal  $((SX)^T SX)^{-1} (SX)^T SY$ , since the inverse will *not* exist if the sketching process does not preserve rank.) Our goal here is to compare the performance of  $\beta_S$  to the

$\beta_{OLS}$ . We will do so by considering three related performance criteria, two of a statistical flavor, and one of a more algorithmic or worst-case flavor.

From a statistical perspective, it is common to assume a standard linear model on  $Y$ ,

$$Y = X\beta + \epsilon,$$

where we remind the reader that  $\beta \in \mathbb{R}^p$  is the true parameter and  $\epsilon \in \mathbb{R}^n$  is a standardized noise vector, with  $\mathbb{E}[\epsilon] = 0$  and  $\mathbb{E}[\epsilon\epsilon^T] = I_{n \times n}$ . From this statistical perspective, we will consider the following two criteria.

- The first statistical criterion we consider is the relative *statistical prediction efficiency* (SPE), defined as follows:

$$C_{SPE}(S) = \frac{\mathbb{E}[\|X(\beta - \beta_S)\|_2^2]}{\mathbb{E}[\|X(\beta - \beta_{OLS})\|_2^2]}, \quad (7)$$

where the expectation  $\mathbb{E}[\cdot]$  is taken over the random noise  $\epsilon$ .

- The second statistical criterion we consider is the relative *statistical residual efficiency* (SRE), defined as follows:

$$C_{SRE}(S) = \frac{\mathbb{E}[\|Y - X\beta_S\|_2^2]}{\mathbb{E}[\|Y - X\beta_{OLS}\|_2^2]}, \quad (8)$$

where, again, the expectation  $\mathbb{E}[\cdot]$  is taken over the random noise  $\epsilon$ .

Recall that the standard relative statistical efficiency for two estimators  $\beta_1$  and  $\beta_2$  is defined as  $\text{eff}(\beta_1, \beta_2) = \frac{\text{Var}(\beta_2)}{\text{Var}(\beta_1)}$ , where  $\text{Var}(\cdot)$  denotes the variance of the estimator (see e.g. [17]). For the SPE, we have replaced the variance of each estimator by the mean-squared prediction error. For the SRE, we use the term residual since for any estimator  $\hat{\beta}$ ,  $Y - X\hat{\beta}$  are the residuals for estimating  $Y$ .

From an algorithmic perspective, there is no noise process  $\epsilon$ . Instead,  $X$  and  $Y$  are arbitrary, and  $\beta$  is simply computed from Eqn (5). To draw a parallel with the usual statistical generative process, however, and to understand better the relationship between various objectives, consider “defining”  $Y$  in terms of  $X$  by the following “linear model”:

$$Y = X\beta + \epsilon,$$

where  $\beta \in \mathbb{R}^p$  and  $\epsilon \in \mathbb{R}^n$ . Importantly,  $\beta$  and  $\epsilon$  here represent different quantities than in the usual statistical setting. Rather than  $\epsilon$  representing a noise process and thus  $\beta$  representing a “true parameter” that is observed through a noisy  $Y$ , here in the algorithmic setting, we will take advantage of the rank-nullity theorem in linear algebra to relate  $X$  and  $Y$ .<sup>5</sup> To define a “worst case model”  $Y = X\beta + \epsilon$  for the algorithmic setting, one can view the “noise” process  $\epsilon$  to consist of any vector that lies in the null-space of  $X^T$ . Then, since the choice of  $\beta \in \mathbb{R}^p$  is arbitrary, one can construct any arbitrary or worst-case input data  $Y$ . From this algorithmic case, we will consider the following criterion.

- The algorithmic criterion we consider is the *worst-case error* (WCE), defined as follows:

$$C_{WCE}(S) = \sup_Y \frac{\|Y - X\beta_S\|_2^2}{\|Y - X\beta_{OLS}\|_2^2}. \quad (9)$$

<sup>5</sup>The rank-nullity theorem asserts that given any matrix  $X \in \mathbb{R}^{n \times p}$  and vector  $Y \in \mathbb{R}^n$ , there exists a unique decomposition  $Y = X\beta + \epsilon$ , where  $\beta$  is the projection of  $Y$  on to the range space of  $X^T$  and  $\epsilon = Y - X\beta$  lies in the null-space of  $X^T$  [24].

This criterion is worst-case since we take a supremum  $Y$ , and it is the performance criterion that is analyzed in Drineas et al. [8] and Drineas et al. [7], as bounded in Eqn. (3).

Writing  $Y$  as  $X\beta + \epsilon$ , where  $X^T\epsilon = 0$ , the worst-case error can be re-expressed as:

$$C_{WCE}(S) = \sup_{Y=X\beta+\epsilon, X^T\epsilon=0} \frac{\|Y - X\beta_S\|_2^2}{\|Y - X\beta_{OLS}\|_2^2}.$$

Hence, in the worst-case algorithmic setup, we take a supremum over  $\epsilon$ , where  $X^T\epsilon = 0$ , whereas in the statistical setup, we take an expectation over  $\epsilon$  where  $\mathbb{E}[\epsilon] = 0$ .

Before proceeding, several other comments about this algorithmic-statistical framework and our objectives are worth mentioning.

- From the perspective of our two linear models, we have that  $\beta_{OLS} = \beta + (X^T X)^{-1} X^T \epsilon$ . In the statistical setting, since  $\mathbb{E}[\epsilon\epsilon^T] = I_{n \times n}$ , it follows that  $\beta_{OLS}$  is a random variable with  $\mathbb{E}[\beta_{OLS}] = \beta$  and  $\mathbb{E}[(\beta - \beta_{OLS})(\beta - \beta_{OLS})^T] = (X^T X)^{-1}$ . In the algorithmic setting, on the other hand, since  $X^T\epsilon = 0$ , it follows that  $\beta_{OLS} = \beta$ .
- $C_{SRE}(S)$  is a statistical analogue of the worst-case algorithmic objective  $C_{WCE}(S)$ , since both consider the ratio of the metrics  $\frac{\|Y - X\beta_S\|_2^2}{\|Y - X\beta_{OLS}\|_2^2}$ . The difference is that a sup over  $Y$  in the algorithmic setting is replaced by an expectation over noise  $\epsilon$  in the statistical setting. A natural question is whether there is an algorithmic analogue of  $C_{SPE}(S)$ . Such a performance metric would be:

$$\sup_Y \frac{\|X(\beta - \beta_S)\|_2^2}{\|X(\beta - \beta_{OLS})\|_2^2}, \quad (10)$$

where  $\beta$  is the projection of  $Y$  on to the range space of  $X^T$ . However, since  $\beta_{OLS} = \beta + (X^T X)^{-1} X^T \epsilon$  and since  $X^T\epsilon = 0$ ,  $\beta_{OLS} = \beta$  in the algorithmic setting, the denominator of Eqn. (10) equals zero, and thus the objective in Eqn. (10) is not well-defined. The ‘‘difficulty’’ of computing or approximating this objective parallels our results below that show that approximating  $C_{SPE}(S)$  is much more challenging (in terms of the number of samples needed) than approximating  $C_{SRE}(S)$ .

- In the algorithmic setting, the sketching matrix  $S$  and the objective  $C_{WCE}(S)$  can depend on  $X$  and  $Y$  in any arbitrary way, but in the following we consider only sketching matrices that are either independent of both  $X$  and  $Y$  or depend only on  $X$  (e.g., via the statistical leverage scores of  $X$ ). In the statistical setting,  $S$  is allowed to depend on  $X$ , but not on  $Y$ , as any dependence of  $S$  on  $Y$  might introduce correlation between the sketching matrix and the noise variable  $\epsilon$ . Removing this restriction is of interest, especially in light of the recent results of Jia et al. [15] that show that one can obtain WCE bounds of the form Eqn. (3) by constructing  $S$  by randomly sampling according to an importance sampling distribution that depends on the *influence scores*—essentially the leverage scores of the matrix  $X$  augmented with  $-Y$  as an additional column—of the  $(X, Y)$  pair.
- Both  $C_{SPE}(S)$  and  $C_{SRE}(S)$  are qualitatively related to quantities analyzed by Ma et al. [18, 19]. In addition,  $C_{WCE}(S)$  is qualitatively similar to  $\text{Cov}(\hat{\beta}|Y)$  in Ma et al., since in the algorithmic setting  $Y$  is treated as fixed; and  $C_{SRE}(S)$  is qualitatively similar to  $\text{Cov}(\hat{\beta})$  in Ma et al., since in the statistical setting  $Y$  is treated as random and coming from a linear model. That being said, the metrics and results we present in this paper are not directly comparable to those of Ma et al. since, e.g., they had a slightly different setup than we have here, and since they used a first-order Taylor approximation while we do not.

## 2.2 Structural results on sketching matrices

We are now ready to develop structural conditions characterizing how the sketching matrix  $S$  interacts with the data matrix  $X$  that will allow us to provide upper bounds for the quantities  $C_{WCE}(S)$ ,  $C_{SPE}(S)$ , and  $C_{SRE}(S)$ . To do so, recall that given the data matrix  $X$ , we can express the singular value decomposition of  $X$  as  $X = U\Sigma V^T$ , where  $U \in \mathbb{R}^{n \times p}$  is an orthogonal matrix, i.e.,  $U^T U = I_{p \times p}$ . In addition, we can define the *oblique projection* matrix

$$\Pi_S^U := U(SU)^\dagger S. \quad (11)$$

Note that if  $\text{rank}(SX) = p$ , then  $\Pi_S^U$  can be expressed as  $\Pi_S^U = U(U^T S^T S U)^{-1} U^T S^T S$ , since  $U^T S^T S U$  is invertible. Importantly however, depending on the properties of  $X$  and how  $S$  is constructed, it can easily happen that  $\text{rank}(SX) < p$ , even if  $\text{rank}(X) = p$ .

Given this setup, we can now state the following lemma, the proof of which may be found in Section 5.1. This lemma characterizes how  $C_{WCE}(S)$ ,  $C_{SPE}(S)$ , and  $C_{SRE}(S)$  depend on different structural properties of  $\Pi_S^U$  and  $SU$ .

**Lemma 1.** *For the algorithmic setting,*

$$C_{WCE}(S) = 1 + \sup_{\delta \in \mathbb{R}^p, U^T \epsilon = 0} \left[ \frac{\|(I_{p \times p} - (SU)^\dagger (SU))\delta\|_2^2}{\|\epsilon\|_2^2} + \frac{\|\Pi_S^U \epsilon\|_2^2}{\|\epsilon\|_2^2} \right].$$

*For the statistical setting,*

$$C_{SPE}(S) = \frac{\|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T \beta\|_2^2}{p} + \frac{\|\Pi_S^U\|_F^2}{p},$$

and

$$C_{SRE}(S) = 1 + \frac{\|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T \beta\|_2^2}{n - p} + \frac{\|\Pi_S^U\|_F^2 - p}{n - p} = 1 + \frac{C_{SPE}(S) - 1}{n/p - 1}.$$

Several points are worth making about Lemma 1.

- For all 3 criteria, the term which involves  $(SU)^\dagger SU$  is a “bias” term that is non-zero in the case that  $\text{rank}(SU) < p$ . For  $C_{SPE}(S)$  and  $C_{SRE}(S)$ , the term corresponds exactly to the statistical bias; and if  $\text{rank}(SU) = p$ , meaning that  $S$  is a *rank-preserving* sketching matrix, then the bias term equals 0, since  $(SU)^\dagger SU = I_{p \times p}$ . In practice, if  $r$  is chosen smaller than  $p$  or larger than but very close to  $p$ , it may happen that  $\text{rank}(SU) < p$ , in which case this bias is incurred.
- The final equality  $C_{SRE}(S) = 1 + \frac{C_{SPE}(S) - 1}{n/p - 1}$  shows that in general it is much more difficult (in terms of the number of samples needed) to obtain bounds on  $C_{SPE}(S)$  than  $C_{SRE}(S)$ —since  $C_{SRE}(S)$  re-scales  $C_{SPE}(S)$  by  $p/n$ , which is much less than 1. This will be reflected in the main results below, where the scaling of  $C_{SRE}(S)$  will be a factor of  $p/n$  smaller than  $C_{SPE}(S)$ . In general, it is significantly more difficult to bound  $C_{SPE}(S)$ , since  $\|X(\beta - \beta_{OLS})\|_2^2$  is  $p$ , whereas  $\|Y - X\beta_{OLS}\|_2^2$  is  $n - p$ , and so there is much less margin for error in approximating  $C_{SPE}(S)$ .
- In the algorithmic or worst-case setting,  $\sup_{\epsilon \in \mathbb{R}^n / \{0\}, U^T \epsilon = 0} \frac{\|\Pi_S^U \epsilon\|_2^2}{\|\epsilon\|_2^2}$  is the relevant quantity, whereas in the statistical setting  $\|\Pi_S^U\|_F^2$  is the relevant quantity. The Frobenius norm enters in the statistical setting because we are taking an average over homoscedastic noise, and so the  $\ell_2$  norm of the eigenvalues of  $\Pi_S^U$  need to be controlled. On the other hand, in the algorithmic or worst-case setting, the worst direction in the null-space of  $U^T$  needs to be controlled, and thus the spectral norm enters.



In order to provide a convenient way to parameterize our upper bounds for  $C_{WCE}(S)$ ,  $C_{SPE}(S)$ , and  $C_{SRE}(S)$ , we introduce the following three structural conditions on  $S$ . Let  $\tilde{\sigma}_{\min}(A)$  denote the minimum *non-zero* singular value of a matrix  $A$ .

- The first condition is that there exists an  $\alpha(S) > 0$  such that

$$\tilde{\sigma}_{\min}(SU) \geq \alpha(S). \quad (12)$$

- The second condition is that there exists a  $\beta(S)$  such that

$$\sup_{\epsilon, U^T \epsilon = 0} \frac{\|U^T S^T S \epsilon\|_2}{\|\epsilon\|_2} \leq \beta(S). \quad (13)$$

- The third condition is that there exists a  $\gamma(S)$  such that

$$\|U^T S^T S\|_F \leq \gamma(S). \quad (14)$$

Note that the structural conditions defined by  $\alpha(S)$  and  $\beta(S)$  have been defined previously as Eqn. (8) and Eqn. (9) in Drineas et al. [8].

Given these quantities, we can state the following lemma, the proof of which may be found in Section 5.2. This lemma provides upper bounds for  $C_{WCE}(S)$ ,  $C_{SPE}(S)$ , and  $C_{SRE}(S)$  in terms of the parameters  $\alpha(S)$ ,  $\beta(S)$ , and  $\gamma(S)$ .

**Lemma 2.** *For  $\alpha(S)$  and  $\beta(S)$ , as defined in Eqn. (12) and (13),*

$$C_{WCE}(S) \leq 1 + \sup_{\delta \in \mathbb{R}^p, U^T \delta = 0} \frac{\|(I_{p \times p} - (SU)^\dagger (SU)) \delta\|_2^2}{\|\delta\|_2^2} + \frac{\beta^2(S)}{\alpha^4(S)}.$$

*For  $\alpha(S)$  and  $\gamma(S)$ , as defined in Eqn. (12) and (14),*

$$C_{SPE}(S) \leq \frac{\|(I_{p \times p} - (SU)^\dagger SU) \Sigma V^T \beta\|_2^2}{p} + \frac{\gamma^2(S)}{\alpha^4(S)}.$$

*Furthermore,*

$$C_{SRE}(S) \leq 1 + \frac{p}{n} \left[ \frac{\|(I_{p \times p} - (SU)^\dagger SU) \Sigma V^T \beta\|_2^2}{p} + \frac{\gamma^2(S)}{\alpha^4(S)} \right].$$

Again, the terms involving  $(SU)^\dagger SU$  are a “bias” that equal zero for rank-preserving sketching matrices. In addition, we emphasize that the results of Lemma 1 and Lemma 2 hold for arbitrary sketching matrices  $S$ . In the following, we bound  $\alpha(S)$ ,  $\beta(S)$  and  $\gamma(S)$  for several different randomized sketching matrices, and this will permit us to obtain bounds on  $C_{WCE}(S)$ ,  $C_{SPE}(S)$ , and  $C_{SRE}(S)$ . For the sketching matrices we analyze for the remainder of the paper, we prove that the bias term is 0 with high probability.

### 3 Main theoretical results

In this section, we provide upper bounds for  $C_{WCE}(S)$ ,  $C_{SPE}(S)$ , and  $C_{SRE}(S)$ , where  $S$  correspond to random sampling and random projection matrices. In particular, we provide upper bounds for 4 sketching matrices: (1) a vanilla leverage-based random sampling algorithm from Drineas et al. [7]; (2) a variant of leverage-based random sampling, where the random samples are *not* re-scaled prior to their inclusion in the sketch; (3) a vanilla random projection algorithm, where  $S$  is a random matrix containing i.i.d. sub-Gaussian random variables; and (4) a random projection algorithm, where  $S$  is a random Hadamard-based random projection, as analyzed in [2].

### 3.1 Random sampling methods

Here, we consider random sampling algorithms. To do so, first define a random sampling matrix  $\tilde{S} \in \mathbb{R}^n$  as follows:  $\tilde{S}_{ij} \in \{0, 1\}$  for all  $(i, j)$  and  $\sum_{j=1}^n \tilde{S}_{ij} = 1$ , where each row has an independent multinomial distribution with probabilities  $(p_i)_{i=1}^n$ . The matrix of cross-leverage scores is defined as  $L = UU^T \in \mathbb{R}^{n \times n}$ , and  $\ell_i = L_{ii}$  denotes the leverage score corresponding to the  $i^{\text{th}}$  sample. Note that the leverage scores satisfy  $\sum_{i=1}^n \ell_i = \text{trace}(L) = p$  and  $0 \leq \ell_i \leq 1$ . The sampling probability distribution we consider  $(p_i)_{i=1}^n$  is of the form  $p_i = (1 - \theta)\frac{\ell_i}{p} + \theta\frac{1}{n}$ , where  $0 \leq \theta < 1$ , i.e., it is a convex combination of a leverage-based distribution and the uniform distribution. Note that for  $\theta = 0$ , the probabilities are proportional to the leverage scores, whereas for  $\theta = 1$ , the probabilities are uniform.

We consider two sampling matrices, one where the random sampling matrix is re-scaled, as in Drineas et al. [8], and one in which no re-scaling takes place. In particular, let  $S_{NR} = \tilde{S}$  denote the random sampling matrix (where the subscript  $NR$  denotes the fact that no re-scaling takes place). The re-scaled sampling matrix is  $S_R \in \mathbb{R}^{r \times n} = \tilde{S}W$ , where  $W \in \mathbb{R}^{n \times n}$  is a diagonal re-scaling matrix, where  $[W]_{jj} = \sqrt{\frac{1}{rp_j}}$  and  $W_{ji} = 0$  for  $j \neq i$ . The quantity  $\frac{1}{p_j}$  is the re-scaling factor. In this case, we have the following result, the proof of which may be found in Section 5.3.

**Theorem 1.** *For  $S = S_R$ , with  $r \geq \frac{Cp}{(1-\theta)} \log\left(\frac{C'p}{(1-\theta)}\right)$ , then with probability at least 0.7, it holds that  $\text{rank}(S_R U) = p$  and that:*

$$\begin{aligned}\alpha^2(S_R) &\geq \frac{1}{\sqrt{2(1-\theta)}} \\ \beta(S_R) &\leq \sqrt{11\frac{p}{(1-\theta)r}} \\ \gamma(S_R) &\leq \sqrt{11\frac{n}{r}}.\end{aligned}$$

*In this case, it holds that:*

$$\begin{aligned}C_{WCE}(S_R) &\leq 1 + 12\frac{p}{(1-\theta)r} \\ C_{SPE}(S_R) &\leq 44\frac{n}{(1-\theta)r} \\ C_{SRE}(S_R) &\leq 1 + 44\frac{p}{(1-\theta)r}.\end{aligned}$$

Several things are worth noting about this result. First, note that both  $C_{WCE}(S_R) - 1$  and  $C_{SRE}(S_R) - 1$  scale as  $\frac{p}{r}$ ; thus, it is possible to obtain high-quality performance guarantees for ordinary least squares, as long as  $\frac{p}{r} \rightarrow 0$ , e.g., if  $r$  is only slightly larger than  $p$ . On the other hand,  $C_{SPE}(S_R)$  scales as  $\frac{n}{r}$ , meaning  $r$  needs to be close to  $n$  to provide similar performance guarantees. Next, note that all of the upper bounds apply to any data matrix  $X$ , without assuming any additional structure on  $X$ . Finally, note that when  $\theta = 1$ , which corresponds to uniform sampling of the rows, all the upper bounds blow up. Our simulations also reveal that uniform sampling generally performs more poorly than leverage-score based approaches under the linear models we consider.

Next, we consider the leverage-score estimator without re-scaling  $S_{NR}$ . In order to develop nontrivial bounds on  $C_{WCE}(S_{NR})$ ,  $C_{SPE}(S_{NR})$ , and  $C_{SRE}(S_{NR})$ , we need to make a strong assumption on the leverage-score distribution on  $X$ . To do so, we define the following.

**Definition 1** ( $k$ -heavy hitter leverage distribution). A sequence of leverage scores  $(\ell_i)_{i=1}^n$  is a  $k$ -heavy hitter leverage score distribution if there exist constants  $c, C > 0$  such that for  $1 \leq i \leq k$ ,  $\frac{cp}{k} \leq \ell_i \leq \frac{Cp}{k}$  and for the remaining  $n - k$  leverage scores,  $\sum_{i=k+1}^p \ell_i \leq \frac{3}{4}$ .

The interpretation of a  $k$ -heavy hitter leverage distribution is one in which only  $k$  samples in  $X$  contain the majority of the leverage score mass. In the simulations below, we provide examples of synthetic matrices  $X$  where the majority of the mass is in the largest leverage scores. The parameter  $k$  acts as a measure of non-uniformity, in that the smaller the  $k$ , the more non-uniform are the leverage scores. The  $k$ -heavy hitter leverage distribution allows us to model highly non-uniform leverage scores. In this case, we have the following result, the proof of which may be found in Section 5.4.

**Theorem 2.** For  $S = S_{NR}$ , with  $\theta = 0$  and assuming a  $k$ -heavy hitter leverage distribution and  $r \geq c_1 p \log(c_2 p)$ , then with probability at least 0.6, it holds that  $\text{rank}(S_{NR}) = p$  and that:

$$\begin{aligned}\alpha^2(S_{NR}) &\geq \frac{cr}{4Ck} \\ \beta(S_{NR}) &\leq \sqrt{11} p \frac{r}{k} \\ \gamma(S_{NR}) &\leq \sqrt{11C^2 \frac{r}{k}}.\end{aligned}$$

In this case, it holds that:

$$\begin{aligned}C_{WCE}(S_{NR}) &\leq 1 + \frac{44C^2 p}{c^2 r} \\ C_{SPE}(S_{NR}) &\leq \frac{44C^4 k}{c^2 r} \\ C_{SRE}(S_{NR}) &\leq 1 + \frac{44C^4 pk}{c^2 nr}.\end{aligned}$$

Notice that when  $k \ll n$ , bounds in Theorem 2 on  $C_{SPE}(S_{NR})$  and  $C_{SRE}(S_{NR})$  are significantly sharper than bounds in Theorem 1 on  $C_{SPE}(S_R)$  and  $C_{SRE}(S_R)$ . Hence not re-scaling has the potential to provide sharper bound in the statistical setting. However a stronger assumption on  $X$  is needed for this result.

### 3.2 Random projection methods

Here, we consider two random projection algorithms, one based on a sub-Gaussian projection matrix and the other based on a Hadamard projection matrix. To do so, define  $[S_{SGP}]_{ij} = \frac{1}{\sqrt{r}} X_{ij}$ , where  $(X_{ij})_{1 \leq i \leq r, 1 \leq j \leq n}$  are i.i.d. sub-Gaussian random variables with  $\mathbb{E}[X_{ij}] = 0$ , variance  $\mathbb{E}[X_{ij}^2] = \sigma^2$  and sub-Gaussian parameter 1. In this case, we have the following result, the proof of which may be found in Section 5.5.

**Theorem 3.** For any matrix  $X$ , there exists a constant  $c$  such that if  $r \geq c' \log n$ , then with probability greater than 0.7, it holds that  $\text{rank}(S_{SGP}) = p$  and that:

$$\begin{aligned}\alpha^2(S_{SGP}) &\geq \frac{1}{\sqrt{2}} \\ \beta(S_{SGP}) &\leq \sqrt{11 \frac{p}{r}} \\ \gamma(S_{SGP}) &\leq \sqrt{11(1 + \frac{n}{r})}.\end{aligned}$$

In this case, it holds that:

$$\begin{aligned} C_{WCE}(S_{SGP}) &\leq 1 + 11\frac{p}{r} \\ C_{SPE}(S_{SGP}) &\leq 44(1 + \frac{n}{r}) \\ C_{SRE}(S_{SGP}) &\leq 1 + 44\frac{p}{r}. \end{aligned}$$

Notice that the bounds in Theorem 3 for  $S_{SGP}$  are equivalent to the bounds in Theorem 1 for  $S_R$ , except that  $r$  is required only to be larger than  $O(\log n)$  rather than  $O(p \log p)$ . Hence for smaller values of  $p$ , random sub-Gaussian projections are more stable than leverage-score sampling based approaches. This reflects the fact that to a first-order approximation, leverage-score sampling performs as well as performing a smooth projection.

Next, we consider the randomized Hadamard projection matrix. In particular,  $S_{Had} = S_{Unif}HD$ , where  $H \in \mathbb{R}^{n \times n}$  is the standard Hadamard matrix (see e.g. [12]),  $S_{Unif} \in \mathbb{R}^{r \times n}$  is an  $r \times n$  uniform sampling matrix, and  $D \in \mathbb{R}^{n \times n}$  is a diagonal matrix with random equiprobable  $\pm 1$  entries. In this case, we have the following result, the proof of which may be found in Section 5.6.

**Theorem 4.** *For any matrix  $X$ , there exists a constant  $c$  such that if  $r \geq cp \log n(\log p + \log \log n)$ , then with probability greater than 0.8, it holds that  $\text{rank}(S_{Had}) = p$  and that:*

$$\begin{aligned} \alpha^2(S_{Had}) &\geq \frac{1}{\sqrt{2}} \\ \beta(S_{Had}) &\leq \sqrt{20 \log(np) \frac{p}{r}} \\ \gamma(S_{Had}) &\leq \sqrt{20 \log(np) (1 + \frac{n}{r})}. \end{aligned}$$

In this case, it holds that:

$$\begin{aligned} C_{WCE}(S_{Had}) &\leq 1 + 40 \log(np) \frac{p}{r} \\ C_{SRE}(S_{Had}) &\leq 40 \log(np) (1 + \frac{n}{r}) \\ C_{SPE}(S_{Had}) &\leq 1 + 40 \log(np) (1 + \frac{p}{r}). \end{aligned}$$

Notice that the bounds in Theorem 4 for  $S_{Had}$  are equivalent to the bounds in Theorem 1 for  $S_R$ , up to a constant and  $\log(np)$  factor. As discussed in Drineas et al. [8], the Hadamard transformation makes the leverage scores of  $X$  approximately uniform (up to  $\log(np)$  factor), which is why the performance is similar to the sub-Gaussian projection (which also tends to make the leverage scores of  $X$  approximately uniform). We suspect that the additional  $\log(np)$  factor is an artifact of the analysis since we use an entry-wise concentration bound; using more sophisticated techniques, we believe that the  $\log(np)$  can be removed.

## 4 Empirical results

In this section, we present the results of an empirical evaluation, illustrating the results of our theory. We will compare the following 6 sketching matrices.

- (1)  $S = S_R$  - random leverage-score sampling with re-scaling.
- (2)  $S = S_{NR}$  - random leverage-score sampling without re-scaling.
- (3)  $S = S_{Unif}$  - random uniform sampling (each sample drawn independently with probability  $1/n$ ).
- (4)  $S = S_{Shr}$  - random leverage-score sampling with re-scaling and with  $\theta = 0.1$ .
- (5)  $S = S_{GP}$  - Gaussian projection matrices.
- (6)  $S = S_{Had}$  - Hadamard projections.

To compare the methods and see how they perform on inputs with different leverage scores, we generate test matrices using a method outlined in Ma et al. [18, 19]. Set  $n = 1024$  (to ensure, for simplicity, an integer power of 2 for the Hadamard transform) and  $p = 50$ , and let the number of samples drawn with replacement,  $r$ , be varied.  $X$  is then generated based on a t-distribution with different choices of  $\nu$  to reflect different uniformity of leverage scores. Each row of  $X$  is selected independently with distribution  $X_i \sim t_\nu(\Sigma)$ , where  $\Sigma$  corresponds to an auto-regressive model with  $\nu$  the degrees of freedom. The 3 values of  $\nu$  presented here are  $\nu = 1$  (highly non-uniform),  $\nu = 2$  (moderately non-uniform), and  $\nu = 10$  (very uniform). See Figure 1 for a plot to see how  $\nu$  influences the uniformity of the leverage scores. For each

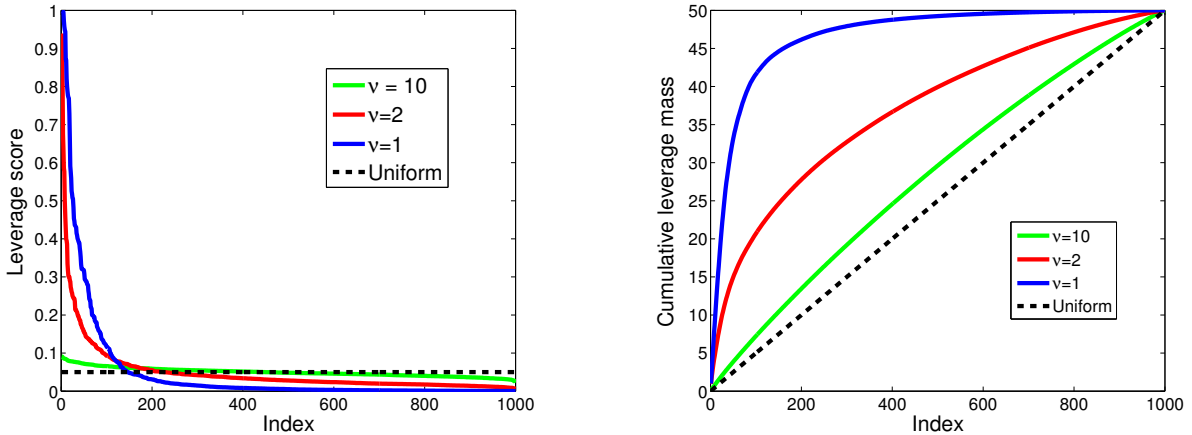


Figure 1: Ordered leverage scores for different values of  $\nu$  (a) and cumulative sum of ordered leverage scores for different values of  $\nu$  (b).

setting, the simulation is repeated 100 times in order to average over both the randomness in the sampling, and in the statistical setting, the randomness over  $y$ .

Note that a natural comparison can be drawn between the parameter  $\nu$  and the parameter  $k$  in the  $k$ -heavy hitter definition. If we want to find the value  $k$  such that 90% of the leverage mass is captured, for  $\nu = 1$ ,  $k \approx 100$ , for  $\nu = 2$ ,  $k \approx 700$  and for  $\nu = 10$ ,  $k \approx 900$ , according to Figure 1 (b). Hence the smaller  $\nu$ , the smaller  $k$  since the leverage-scores are more non-uniform.

We first compare the sketching methods in the statistical setting by comparing  $C_{SPE}(S)$ . In Figure 2, we plot the average  $C_{SPE}(S)$  for the 6 subsampling approaches outlined above, averaged over 100 samples for larger values of  $r$  between 300 and 1000. In addition, in Figure 3, we include a table for results on

smaller values of  $r$  between 80 and 200, to get a sense of the performance when  $r$  is close to  $p$ . Observe that in the large  $r$  setting,  $S_{NR}$  is clearly the best approach, out-performing  $S_R$ , especially for  $\nu = 1$ . For small  $r$ , projection-based methods work better, especially for  $\nu = 1$ , since they tend to uniformize the leverage scores. In addition,  $S_{Shr}$  is superior compared to  $S_{NR}$ , when  $r$  is small, especially when  $\nu = 1$ .

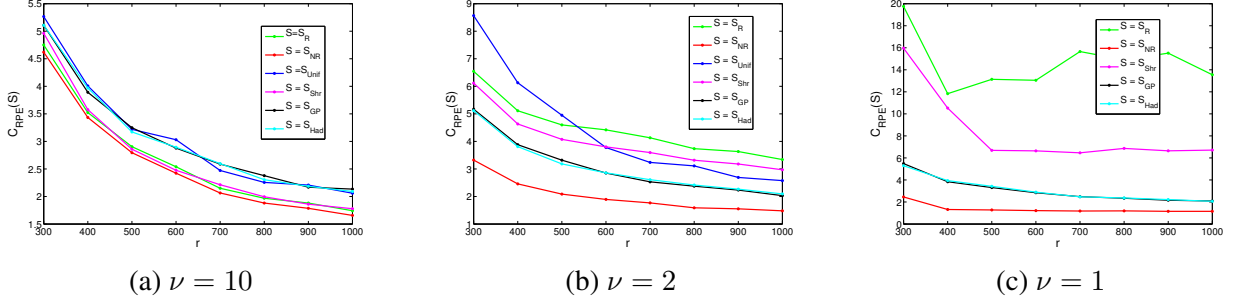


Figure 2: Relative prediction efficiency  $C_{SPE}(S)$  for large  $r$ .

$r$	$S_R$	$S_{NR}$	$S_{Unif}$	$S_{Shr}$	$S_{GP}$	$S_{Had}$
80	39.8	38.7	37.3	39.8	35.5	40.0
90	27.8	27.2	27.2	27.2	26.1	27.4
100	22.4	22.1	22.7	22.2	21.3	23.1
200	8.33	7.88	8.51	7.62	8.20	8.14

(a)  $\nu = 10$

$r$	$S_R$	$S_{NR}$	$S_{Unif}$	$S_{Shr}$	$S_{GP}$	$S_{Had}$
80	70.7	60.1	$1.05 \times 10^2$	73.3	36.5	39.8
90	45.6	34.6	66.2	44.7	26.7	28.2
100	35.1	25.9	52.8	33.8	22.1	22.5
200	9.82	5.54	15.3	9.17	7.59	7.81

(b)  $\nu = 2$

$r$	$S_R$	$S_{NR}$	$S_{Unif}$	$S_{Shr}$	$S_{GP}$	$S_{Had}$
80	$4.4 \times 10^4$	$3.1 \times 10^4$	$7.0 \times 10^3$	$1.4 \times 10^4$	34.2	40.0
90	$1.5 \times 10^4$	$7.0 \times 10^3$	$5.2 \times 10^3$	$1.0 \times 10^4$	26.0	28.7
100	$1.8 \times 10^4$	$3.6 \times 10^3$	$3.9 \times 10^3$	$3.4 \times 10^3$	22.7	24.8
200	$2.0 \times 10^2$	34.0	$5.2 \times 10^2$	$3.6 \times 10^2$	7.94	7.84

(c)  $\nu = 1$

Figure 3: Relative prediction efficiency  $C_{SPE}(S)$  for small  $r$ .

Overall,  $S_{NR}$ ,  $S_R$ , and  $S_{Shr}$  compare very favorably to  $S_{Unif}$ , which is consistent with Theorem 2, since samples with higher leverage scores tend to reduce the mean-squared error. Furthermore,  $S_R$  (which recall involves re-scaling) only increases the mean-squared error, which is again consistent with the theoretical results. The effects are more apparent as the leverage score distribution is more non-uniform (i.e., for  $\nu = 1$ ).

The theoretical upper bound in Theorems 1-4 suggests that  $C_{SPE}(S)$  is of the order  $\frac{n}{r}$ , independent of the leverage scores of  $X$ , for  $S = S_R$  as well as  $S = S_{Had}$  and  $S_{GP}$ . On the other hand, the simulations suggest that for highly non-uniform leverage scores,  $C_{SPE}(S_R)$  is higher than when the leverage scores are uniform, whereas for  $S = S_{Had}$  and  $S_{GP}$ , the non-uniformity of the leverage scores does not significantly affect the bounds. The reason that  $S_{Had}$  and  $S_{GP}$  are not significantly affected by the leverage-score distribution is that the Hadamard and Gaussian projection has the effect of making the leverage scores of any matrix uniform [8]. The reason for the apparent disparity when  $S = S_R$  is that the theoretical bounds use Markov's inequality which is a crude concentration bound. We suspect that a more refined analysis involving the bounded difference inequality would reflect that non-uniform leverage scores result in a larger  $C_{SPE}(S_R)$ .

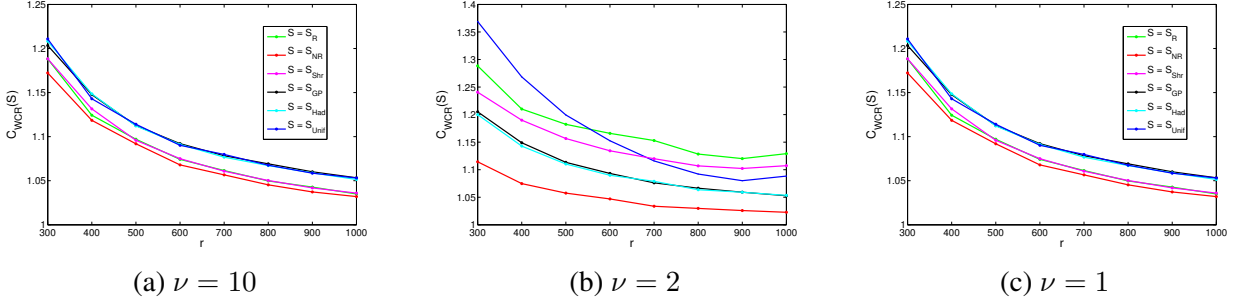


Figure 4: Worst-case relative error  $C_{WCE}(S)$  for large  $r$ .

$r$	$S_R$	$S_{NR}$	$S_{Unif}$	$S_{Shr}$	$S_{GP}$	$S_{Had}$
80	2.82	2.78	2.94	2.82	2.74	2.89
90	2.34	2.32	2.40	2.37	2.24	2.33
100	2.04	2.01	2.09	2.06	2.02	2.03
200	1.33	1.31	1.36	1.32	1.34	1.34

(a)  $\nu = 10$

$r$	$S_R$	$S_{NR}$	$S_{Unif}$	$S_{Shr}$	$S_{GP}$	$S_{Had}$
80	4.46	3.69	5.71	4.33	2.81	2.99
90	3.25	2.85	4.29	3.18	2.27	2.28
100	2.70	2.20	3.52	2.61	2.06	2.10
200	1.43	1.22	1.70	1.42	1.34	1.36

(b)  $\nu = 2$

$r$	$S_R$	$S_{NR}$	$S_{Unif}$	$S_{Shr}$	$S_{GP}$	$S_{Had}$
80	$6.0 \times 10^9$	$6.0 \times 10^{10}$	$2.1 \times 10^5$	$6.9 \times 10^2$	2.64	2.85
90	$1.7 \times 10^5$	$3.7 \times 10^4$	$2.4 \times 10^5$	$5.0 \times 10^2$	2.35	2.35
100	$9.1 \times 10^4$	$4.5 \times 10^4$	$1.3 \times 10^5$	$1.8 \times 10^2$	2.07	2.12
200	$2.1 \times 10^2$	70.0	$1.6 \times 10^4$	18.0	1.34	1.35

(c)  $\nu = 1$

Figure 5: Worst-case relative error  $C_{WCE}(S)$  for large  $r$ .

Finally, Figures 4 and 5 provide a comparison of the worst-case relative error  $C_{WCE}(S)$  for large and small ( $r > 200$  and  $r \leq 200$ , respectively) values of  $r$ . Observe that, in general,  $C_{WCE}(S)$  are much closer to 1 than  $C_{SPE}(S)$  for all choices of  $S$ . This reflects the scaling of  $\frac{p}{n}$  difference between the bounds. Interestingly, Figures 4 and 5 indicates that  $S_{NR}$  still tends to out-perform  $S_R$  in general, however the difference is not as significant as in the statistical setting.

## 5 Proofs of main results

In this section, we provide proofs of our lemmas and theorems. The proof techniques for all of our results are similar, in that we first bound the expectations of  $\alpha(S)$ ,  $\beta(S)$ , and  $\gamma(S)$ , for each  $S$ , and then we apply Markov's inequality to develop high probability bounds.

### 5.1 Proof of Lemma 1

Recall that  $X = U\Sigma V^T$ , where  $U \in \mathbb{R}^{n \times p}$ ,  $\Sigma \in \mathbb{R}^{p \times p}$  and  $V \in \mathbb{R}^{p \times p}$  denote the left singular matrix, diagonal singular value matrix and right singular matrix respectively.

First we show that  $\|Y - X\beta_{OLS}\|_2^2 = \|\epsilon\|_2^2$ . To do so, observe that

$$\|Y - X\beta_{OLS}\|_2^2 = \|Y - U\Sigma V^T \beta_{OLS}\|_2^2,$$

and set  $\delta_{OLS} = \Sigma V^T \beta_{OLS}$ . It follows that  $\delta_{OLS} = U^T Y$ . Hence

$$\|Y - X\beta_{OLS}\|_2^2 = \|Y - \Pi_U Y\|_2^2,$$

where  $\Pi_U = UU^T$ . For every  $Y \in \mathbb{R}^n$ , there exists a unique  $\delta \in \mathbb{R}^p$  and  $\epsilon \in \mathbb{R}^n$  such that  $U^T \epsilon = 0$  and  $Y = U\delta + \epsilon$ . Hence

$$\|Y - X\beta_{OLS}\|_2^2 = \|(I_{n \times n} - \Pi_U)\epsilon\|_2^2 = \|\epsilon\|_2^2,$$

where the final equality holds since  $\Pi_U \epsilon = 0$ .

Now we analyze  $\|Y - X\beta_S\|_2^2$ . Observe that

$$\|Y - X\beta_S\|_2^2 = \|Y - \Pi_U^S Y\|_2^2,$$

where  $\Pi_U^S = U(SU)^\dagger S$ . Since  $Y = U\delta + \epsilon$ , it follows that

$$\begin{aligned} \|Y - X\beta_S\|_2^2 &= \|U(I_{p \times p} - (SU)^\dagger SU)\delta + (I_{n \times n} - \Pi_U^S)\epsilon\|_2^2 \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\delta\|_2^2 + \|(I_{n \times n} - \Pi_U^S)\epsilon\|_2^2 \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\delta\|_2^2 + \|\epsilon\|_2^2 + \|\Pi_U^S \epsilon\|_2^2. \end{aligned}$$

Therefore for all  $Y$ :

$$C_{WCE}(S) = \frac{\|Y - X\beta_S\|_2^2}{\|Y - X\beta_{OLS}\|_2^2} = 1 + \frac{\|(I_{p \times p} - (SU)^\dagger SU)\delta\|_2^2 + \|\Pi_U^S \epsilon\|_2^2}{\|\epsilon\|_2^2},$$

where  $U^T \epsilon = 0$ . Taking a supremum over  $Y$  and consequently over  $\epsilon$  and  $\delta$  completes the proof for  $C_{WCE}(S)$ .

Now we turn to the proof for  $C_{SPE}(S)$ . First note that

$$\mathbb{E}[\|X(\beta_{OLS} - \beta)\|_2^2] = \mathbb{E}[\|UU^T Y - U\Sigma V^T \beta\|_2^2].$$



Under the linear model  $Y = U\Sigma V^T\beta + \epsilon$ ,

$$\mathbb{E}[\|X(\beta_{OLS} - \beta)\|_2^2] = \mathbb{E}[\|\Pi_U\epsilon\|_2^2].$$

Since  $\mathbb{E}[\epsilon\epsilon^T] = I_{n \times n}$ , it follows that

$$\mathbb{E}[\|X(\beta_{OLS} - \beta)\|_2^2] = \mathbb{E}[\|\Pi_U\epsilon\|_2^2] = \|\Pi_U\|_F^2 = p.$$

For  $\beta_S$ , we have that

$$\begin{aligned} \mathbb{E}[\|X(\beta_S - \beta)\|_2^2] &= \mathbb{E}[\|\Pi_S^U Y - U\Sigma V^T\beta\|_2^2] = \mathbb{E}[\|(U(I - (SU)^\dagger SU)\Sigma V^T\beta + \Pi_S^U\epsilon)\|_2^2] \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \mathbb{E}[\|\Pi_S^U\epsilon\|_2^2] \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \|\Pi_S^U\|_F^2. \end{aligned}$$

Hence  $C_{SPE}(S) = 1/p(\|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \|\Pi_S^U\|_F^2)$  as stated.

For  $C_{SRE}(S)$ , the mean-squared error for  $\delta_{OLS}$  and  $\delta_S$  are

$$\begin{aligned} \mathbb{E}[\|Y - X\beta_{OLS}\|_2^2] &= \mathbb{E}[\|(I - \Pi^U)\epsilon\|_2^2] \\ &= \|I - \Pi^U\|_F^2 = n - p, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[\|Y - X\beta_S\|_2^2] &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \mathbb{E}[\|(I - \Pi_S^U)\epsilon\|_2^2] \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \text{trace}((I - \Pi_S)^T(I - \Pi_S)) \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \text{trace}(I) - 2\text{trace}(\Pi_S) + \|\Pi_S\|_F^2 \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + n - 2p + \|\Pi_S\|_F^2 \\ &= \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + n - p + \|\Pi_S\|_F^2 - p. \end{aligned}$$

Hence,

$$\begin{aligned} C_{SRE}(S) &= \frac{n - p + \|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \|\Pi_S\|_F^2 - p}{n - p} \\ &= 1 + \frac{\|(I_{p \times p} - (SU)^\dagger SU)\Sigma V^T\beta\|_2^2 + \|\Pi_S\|_F^2 - p}{n - p} \\ &= 1 + \frac{C_{SPE}(S) - 1}{n/p - 1}. \end{aligned}$$

## 5.2 Proof of Lemma 2

Note that  $\Pi_S^U = U(SU)^\dagger S$ . Let  $\text{rank}(SU) = k < p$ , and the singular value decomposition is  $SU = \tilde{U}\tilde{\Sigma}\tilde{V}^T$ , where  $\tilde{\Sigma} \in \mathbb{R}^{k \times k}$  is a diagonal matrix with non-zero singular values of  $SU$ . Then,

$$\begin{aligned} \frac{\|\Pi_S^U\epsilon\|_2^2}{\|\epsilon\|_2^2} &= \frac{\|U(SU)^\dagger S\epsilon\|_2^2}{\|\epsilon\|_2^2} = \frac{\|(SU)^\dagger S\epsilon\|_2^2}{\|\epsilon\|_2^2} \\ &= \frac{\|\tilde{V}\tilde{\Sigma}^{-1}\tilde{U}^T S\epsilon\|_2^2}{\|\epsilon\|_2^2} \\ &= \frac{\|\tilde{V}\tilde{\Sigma}^{-2}\tilde{V}^T\tilde{V}\tilde{\Sigma}\tilde{U}^T S\epsilon\|_2^2}{\|\epsilon\|_2^2}, \end{aligned}$$

where we have ignored the bias term which remains unchanged. Note that  $\tilde{V}\tilde{\Sigma}^{-2}\tilde{V}^T \succeq \alpha^{-2}(S)I_{p \times p}$  and  $\tilde{V}\tilde{\Sigma}\tilde{U}^T = (SU)^T = U^T S^T$ . Hence,

$$\begin{aligned} \frac{\|\Pi_S^U \epsilon\|_2^2}{\|\epsilon\|_2^2} &= \frac{\|\tilde{V}\tilde{\Sigma}^{-2}\tilde{V}^T\tilde{V}\tilde{\Sigma}\tilde{U}^T S \epsilon\|_2^2}{\|\epsilon\|_2^2} \\ &\leq \frac{\|U^T S^T S \epsilon\|_2^2}{\alpha^4(S)\|\epsilon\|_2^2} \\ &= \frac{\beta^2(S)}{\alpha^4(S)\|\epsilon\|_2^2}, \end{aligned}$$

and the upper bound on  $C_{WCE}(S)$  follows.

Similarly,

$$\|\Pi_S^U\|_F^2 = \|U(SU)^\dagger S\|_F^2 = \|(SU)^\dagger S\|_F^2 = \|\tilde{V}\tilde{\Sigma}^{-2}\tilde{V}^T\tilde{V}\tilde{\Sigma}\tilde{U}^T S\|_F^2 \leq \alpha^{-4}(S)\|U^T S^T S\|_F^2$$

and the upper bound on  $C_{SPE}(S)$  follows.

### 5.3 Proof of Theorem 1

First we bound  $\alpha^2(S_R)$  by using existing results in Drineas et al. [8]. In particular applying Theorem 4 in [8] with  $\beta = 1 - \theta$ ,  $A = U^T$ ,  $\epsilon = \frac{1}{\sqrt{2}}$  and  $\delta = 0.1$  provides the desired lower bound on  $\alpha(S_R)$ .

To upper bound  $\beta(S_R)$ , we first upper bound its expectation, then apply Markov's inequality. Using the result of Table 1 (second row) of [6] with  $\beta = 1 - \theta$ :

$$\mathbb{E}[\|U^T S_R^T S_R \epsilon\|_2^2] \leq \frac{1}{(1-\theta)r} \|U^T\|_F^2 \|\epsilon\|_2^2 = \frac{p}{(1-\theta)r} \|\epsilon\|_2^2.$$

Applying Markov's inequality,

$$\|U^T S_R^T S_R \epsilon\|_2^2 \leq \frac{11p}{(1-\theta)r} \|\epsilon\|_2^2,$$

with probability at least 0.9.

Finally we bound  $\gamma(S_R)$ :

$$\begin{aligned} \frac{1}{p} [\|U^T S_R^T S_R\|_F^2] &= \frac{1}{p} [\text{trace}(U^T (S_R^T S_R)^2 U)] \\ &= \frac{1}{p} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n U_{ij} U_{kj} [(S_R^T S_R)^2]_{ki} \\ &= \frac{1}{p} \sum_{j=1}^p \sum_{i=1}^n U_{ij}^2 [S_R^T S_R]_{ii}^2 \\ &= \frac{1}{p} \sum_{i=1}^n \ell_i [S_R^T S_R]_{ii}^2, \end{aligned}$$

where the second last equality follows since  $[S_R^T S_R]_{ki}^2 = 0$  for  $k \neq i$  and the final equality follows since  $\ell_i = \sum_{j=1}^p U_{ij}^2$ . First we upper bound  $\mathbb{E}[\gamma(S_R)]$  and then apply Markov's inequality. Recall that  $[S_R]_{ki} = \frac{1}{\sqrt{r p_i}} \sigma_{ki}$  where  $\mathbb{P}(\sigma_{ki} = +1) = p_i$ . Then,

$$\begin{aligned}
\frac{1}{p} \sum_{i=1}^n \ell_i \mathbb{E}([S_R^T S_R]_{ii}^2) &= \frac{1}{r^2 p} \sum_{i=1}^n \frac{\ell_i}{p_i^2} \sum_{m=1}^r \sum_{\ell=1}^r \mathbb{E}[\sigma_{mi}^2 \sigma_{\ell i}^2] \\
&= \frac{1}{r^2 p} \sum_{i=1}^n \frac{\ell_i}{p_i^2} \sum_{m=1}^r \sum_{\ell=1}^r \mathbb{E}[\sigma_{mi} \sigma_{\ell i}] \\
&= \frac{1}{r^2 p} \sum_{i=1}^n \frac{\ell_i}{p_i^2} [(r^2 - r)p_i^2 + r p_i] \\
&= \frac{1}{r^2 p} \sum_{i=1}^n (\ell_i (r^2 - r) + r \frac{\ell_i}{p_i}) \\
&= 1 - \frac{1}{r} + \frac{1}{r p} \sum_{i=1}^n \frac{\ell_i}{p_i}.
\end{aligned}$$

Substituting  $p_i = (1 - \theta) \frac{\ell_i}{p} + \frac{\theta}{n}$  completes the upper bound on  $\mathbb{E}[\gamma(S_R)]$ :

$$\begin{aligned}
1 - \frac{1}{r} + \frac{1}{r p} \sum_{i=1}^n \frac{\ell_i}{p_i} &= 1 - \frac{1}{r} + \frac{1}{r p} \sum_{i=1}^n \frac{\ell_i}{(1 - \theta) \frac{\ell_i}{p} + \theta \frac{1}{n}} \\
&= 1 - \frac{1}{r} + \frac{n}{r} \sum_{i=1}^n \frac{1}{(1 - \theta)n + \theta \frac{p}{\ell_i}} \\
&\leq 1 - \frac{1}{r} + \frac{n}{r} \sum_{i=1}^n \min\left(\frac{1}{(1 - \theta)n}, \frac{p}{\theta \ell_i}\right) \\
&\leq 1 - \frac{1}{r} + \frac{n}{r} \min\left(\frac{1}{1 - \theta}, \frac{1}{\theta}\right) \\
&\leq 1 + \frac{(2n - 1)}{r}.
\end{aligned}$$

Using Markov's inequality,

$$\mathbb{P}(|\gamma(S_R) - \mathbb{E}[\gamma(S_R)]| \geq 10 \mathbb{E}[\gamma(S_R)]) \leq 0.1,$$

and consequently  $\gamma(S_R) \leq 11(1 + \frac{2n}{r})$  with probability greater than 0.9. The final probability of 0.7 arises since we simulatenously require all three bounds to hold which hold with porbability  $0.9^3 > 0.7$ .

#### 5.4 Proof of Theorem 2

Define  $\bar{S} = \sqrt{\frac{k}{r}} S_{NR}$  where  $S_{NR}$  is the sampling matrix without re-scaling. Recall the  $k$ -heavy hitter leverage-score assumption. Since  $\sum_{i=k+1}^n \ell_i \leq \frac{p}{10r}$ ,  $\sum_{i=k+1}^n p_i \leq \frac{1}{10r}$  (recall  $p_i = \frac{\ell_i}{p}$ ). Hence the probability that a sample only contains the  $k$  samples with high leverage score is:

$$\left(1 - \frac{1}{10r}\right)^r \geq 1 - \frac{1}{10} = 0.9.$$

For the remainder of the proof, we condition on the event  $\mathcal{A}$  that only the rows with the  $k$  largest leverage scores are selected. Let  $\tilde{U} \in \mathbb{R}^{k \times p}$  be the sub-matrix of  $U$  corresponding to the top  $k$  leverage scores.

Let  $W = \mathbb{E}[\bar{S}^T \bar{S}] \in \mathbb{R}^{k \times k}$ . Since  $\frac{c}{k} \leq p_i \leq \frac{C}{k}$  for all  $1 \leq i \leq k$ ,  $cI_{k \times k} \preceq W \preceq CI_{k \times k}$ . Furthermore since  $\sum_{i=k+1}^n \ell_i \leq \frac{p}{10r}$ ,  $0.9I_{p \times p} \preceq \tilde{U}^T \tilde{U} \preceq I_{p \times p}$ .

First we lower bound  $\alpha^2(S_{NR})$ . Applying Theorem 4 in [8] with  $\beta = C$ ,  $A = \tilde{U}^T W^{1/2}$ ,  $\epsilon = \frac{c}{2}$  and  $\delta = 0.1$  ensures that as long as  $r \geq c'p \log(p)$  for sufficiently large  $c'$ ,

$$\|\tilde{U}^T W \tilde{U} - \tilde{U}^T \bar{S}^T \bar{S} \tilde{U}\|_{op} \leq \frac{c}{2C},$$

with probability at least 0.9. Since  $\tilde{U}^T W \tilde{U} \succeq \frac{3c}{4}$ ,  $\tilde{U}^T \bar{S}^T \bar{S} \tilde{U} \succeq \frac{c}{4}$ . Therefore with probability at least 0.9,

$$\alpha^2(S_{NR}) \geq \frac{cr}{4Ck}.$$

Next we bound  $\beta(S_{NR})$ . Since  $\bar{S} = \sqrt{\frac{k}{r}} S_{NR}$ , if we condition on  $\mathcal{A}$ , only the leading  $k$  leverage scores are selected and let  $\tilde{U} \in \mathbb{R}^{k \times p}$  be the sub-matrix of  $U$  corresponding to the top  $k$  leverage scores. Using the result of Table 1 (second row) of [6] with  $\beta = 1$ :

$$\mathbb{E}[\|U^T S_{NR}^T S_{NR} \epsilon\|_2^2] = \frac{r^2}{k^2} \mathbb{E}[\|U^T \bar{S}^T \bar{S} \epsilon\|_2^2] = \frac{r^2}{k^2} \mathbb{E}[\|\tilde{U}^T \bar{S}^T \bar{S} \epsilon\|_2^2] \leq \frac{r^2}{k^2} \|\tilde{U}^T\|_2^2 \|\epsilon\|_2^2 \leq \frac{pr^2}{k^2} \|\epsilon\|_2^2.$$

Applying Markov's inequality,

$$\|U^T S_{NR}^T S_{NR} \epsilon\|_2^2 \leq \frac{11pr^2}{k^2} \|\epsilon\|_2^2,$$

with probability at least 0.9 which completes the upper bound for  $\beta(S_{NR})$ .

Finally we bound  $\gamma(S_{NR})$ :

$$\begin{aligned} \text{trace}(\tilde{U}^T (\bar{S}^T \bar{S})^2 \tilde{U})/p &= \frac{1}{p} \sum_{i=1}^n [\bar{S}^T \bar{S}]_{ii}^2 \sum_{j=1}^p \tilde{U}_{ij}^2 \\ &= \frac{1}{p} \sum_{i=1}^k \ell_i [\bar{S}^T \bar{S}]_{ii}^2 \\ &\leq \frac{Ck}{r} \sum_{i=1}^k \frac{1}{r} \left( \sum_{m=1}^r \sigma_{mi}^2 \right)^2, \end{aligned}$$

where the last step follows since  $\ell_i \leq \frac{Cp}{k}$  for  $1 \leq i \leq k$  and 0 otherwise. Now taking expectations:

$$\begin{aligned} \frac{1}{r} \mathbb{E} \left[ \sum_{i=1}^k \left( \sum_{m=1}^r \sigma_{mi} \right)^2 \right] &= \frac{1}{r} \sum_{i=1}^k \sum_{\ell=1}^r \sum_{m=1}^r \mathbb{E}[\sigma_{\ell i} \sigma_{m i}] \\ &\leq \frac{C^2}{r} \sum_{i=1}^k \left( \frac{r^2 - r}{k^2} + \frac{r}{k} \right) \\ &= C^2 \left( \frac{r-1}{k} + 1 \right) \\ &\leq C^2 \left( \frac{r}{k} + 1 \right). \end{aligned}$$

Since  $S_{NR} = \sqrt{\frac{k}{r}}\bar{S}$ ,  $\mathbb{E}[\text{trace}(U^T(S_{NR}^T S_{NR})^2 U)/p] \leq C^2(1 + \frac{k}{r})$ . Applying Markov's inequality,

$$\text{trace}(U^T(S_{NR}^T S_{NR})^2 U)/p \leq 11C^2(1 + \frac{k}{r}),$$

with probability at least 0.9. The probability of 0.6 arises since  $0.9^4 > 0.6$ .

### 5.5 Proof of Theorem 3

First we bound the smallest singular value of  $S_{SGP}(U)$ . Using standard results for bounds on the eigenvalues of sub-Gaussian matrices (see Proposition 2.4 in Rudelson and Vershynin [27]), each entry of  $A \in \mathbb{R}^{r \times n}$  is an i.i.d. zero-mean sub-Gaussian matrix:

$$\mathbb{P}(\inf_{\|x\|_2=1} \frac{1}{\sqrt{r}} \|Ax\|_2^2 \leq \frac{1}{\sqrt{2}}) \leq n \exp(-cr).$$

Hence, provided  $cr \geq 2 \log n$ ,

$$\alpha(S_{SGP}) \geq \frac{1}{\sqrt{2}},$$

with probability greater than  $1 - c \exp(-c'r)$ .

Next we bound  $\beta(S_{SGP})$ . Since  $U^T \epsilon = 0$ ,  $\|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2 = \|U^T (S_{SGP}^T S_{SGP} - I_{n \times n}) \epsilon\|_2^2$ . Therefore

$$\begin{aligned} \|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2 &= \sum_{j=1}^p \left( \sum_{i=1}^n \sum_{k=1}^n U_{ij} (S_{SGP}^T S_{SGP} - I_{n \times n})_{ik} \epsilon_k \right)^2 \\ &= \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{\ell=1}^n U_{ij} U_{mj} (S_{SGP}^T S_{SGP} - I_{n \times n})_{ik} (S_{SGP}^T S_{SGP} - I_{n \times n})_{m\ell} \epsilon_k \epsilon_\ell. \end{aligned}$$

First we bound  $\mathbb{E}[\|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2]$ .

$$\mathbb{E}[\|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2] = \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{\ell=1}^n U_{ij} U_{mj} \mathbb{E}[(S_{SGP}^T S_{SGP} - I_{n \times n})_{ik} (S_{SGP}^T S_{SGP} - I_{n \times n})_{m\ell}] \epsilon_k \epsilon_\ell.$$

Recall that  $S_{SGP} \in \mathbb{R}^{r \times n}$ ,  $[S_{SGP}]_{si} = \frac{X_{si}}{\sqrt{r}}$  where  $X_{si}$  are i.i.d. sub-Gaussian random variables with mean 0 and sub-Gaussian parameter 1. Hence

$$\mathbb{E}[(S_{SGP}^T S_{SGP} - I_{n \times n})_{ik} (S_{SGP}^T S_{SGP} - I_{n \times n})_{m\ell}] = \frac{1}{r} (\mathbb{I}(i = k) \mathbb{I}(\ell = m) + \mathbb{E}[X_i X_j X_\ell X_m]),$$

where  $X_i, X_k, X_\ell$  and  $X_m$  are i.i.d. sub-Gaussian random variables. Therefore

$$\mathbb{E}[\|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2] = \frac{1}{r} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{\ell=1}^n U_{ij} U_{mj} (\mathbb{I}(i = k) \mathbb{I}(\ell = m) + \mathbb{E}[X_i X_k X_\ell X_m]) \epsilon_k \epsilon_\ell.$$

First note that  $\mathbb{I}(i = k) \mathbb{I}(\ell = m) + \mathbb{E}[X_i X_k X_\ell X_m] = 0$  unless  $i = k$  and  $\ell = m$ , or  $i = \ell$  and  $k = \ell$  or any other combination of two pairs of variables have the same index. When  $i = k = \ell = m$ ,  $\mathbb{I}(i =$

$k)\mathbb{I}(\ell = m) + \mathbb{E}[X_i X_k X_\ell X_m] = 1 + E[X_i^4] \leq 2$ , since for sub-Gaussian random variables with parameter 1,  $E[X_i^4] \leq 1$  and

$$\begin{aligned}
\frac{1}{r} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{\ell=1}^n U_{ij} U_{mj} (\mathbb{I}(i = k)\mathbb{I}(\ell = m) + \mathbb{E}[X_i X_k X_\ell X_m]) \epsilon_k \epsilon_\ell &= \frac{1}{r} \sum_{j=1}^p \sum_{i=1}^n U_{ij}^2 \epsilon_i^2 (1 + E[X_i^4]) \\
&\leq \frac{2}{r} \sum_{j=1}^p \sum_{i=1}^n U_{ij}^2 \epsilon_i^2 \\
&\leq \frac{2}{r} \sum_{j=1}^p U_{ij}^2 \|\epsilon\|_2^2 \\
&= \frac{2p}{r} \|\epsilon\|_2^2.
\end{aligned}$$

When  $i = k$  and  $\ell = m$  but  $k \neq \ell$ ,  $\mathbb{I}(i = k)\mathbb{I}(\ell = m) + \mathbb{E}[X_i X_k X_\ell X_m] = 2$  and

$$\sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{\ell=1}^n U_{ij} U_{mj} (\mathbb{I}(i = k)\mathbb{I}(\ell = m) + \mathbb{E}[X_i X_k X_\ell X_m]) \epsilon_k \epsilon_\ell = 2 \sum_{j=1}^p \sum_{i=1}^n \sum_{\ell=1}^n U_{ij} U_{mj} \epsilon_i \epsilon_m = 0,$$

since  $U^T \epsilon = 0$ . Using similar logic when the two pairs of variables are not identical, the sum is 0, and hence

$$\mathbb{E}[\|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2] \leq \frac{2p}{r} \|\epsilon\|_2^2,$$

for all  $\epsilon$  such that  $U^T \epsilon = 0$ . Applying Markov's inequality,

$$\|U^T S_{SGP}^T S_{SGP} \epsilon\|_2^2 \leq \frac{22p}{r} \|\epsilon\|_2^2,$$

with probability greater than 0.9. Therefore  $\beta(S_{SGP}) \leq \sqrt{\frac{22p}{r}}$  with probability at least 0.9.

Now we bound  $\gamma(S_{SGP}) = \|U^T S_{SGP}^T S_{SGP}\|_F^2 = \text{trace}(U^T (S_{SGP}^T S_{SGP})^2 U)/p$ :

$$\begin{aligned}
\text{trace}(U^T (S_{SGP}^T S_{SGP})^2 U)/p &= \frac{1}{p} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n [(S_{SGP}^T S_{SGP})^2]_{ik} U_{ij} U_{kj} \\
&= \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{v=1}^n \sum_{m=1}^r \sum_{\ell=1}^r U_{ij} U_{kj} X_{mv} X_{mi} X_{lv} X_{\ell k}.
\end{aligned}$$

First we bound the expectation:

$$\begin{aligned}
\mathbb{E}[\text{trace}(U^T (S_{SGP}^T S_{SGP})^2 U) / p] &= \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{v=1}^n \sum_{m=1}^r \sum_{\ell=1}^r E[X_{mv} X_{mi} X_{\ell v} X_{\ell k}] \\
&= \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n \sum_{v=1}^n (r^2 - r) U_{ij} U_{kj} \mathbb{E}[X_v X_i] \mathbb{E}[X_v X_k] + r \mathbb{E}[X_v^2 X_i X_k] \\
&= \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n (r^2 - r) U_{ij}^2 (\mathbb{E}[X_i^2])^2 + \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n \sum_{v=1}^n r U_{ij}^2 \mathbb{E}[X_v^2 X_i^2] \\
&= \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n (r^2 - r) U_{ij}^2 \sigma^4 + \frac{1}{pr^2} \sum_{j=1}^p \sum_{i=1}^n \sum_{v=1}^n r U_{ij}^2 \mathbb{E}[X_v^2 X_i^2] \\
&= 1 - \frac{1}{r} + \frac{\mu_4}{\sigma^4 r} + \frac{n-1}{r} \\
&\leq 1 + \frac{3}{r} + \frac{n-2}{r} \\
&= 1 + \frac{(n+1)}{r}.
\end{aligned}$$

Applying Markov's inequality,

$$\gamma(S_{SGP}) \leq 11 \left(1 + \frac{(n+1)}{r}\right),$$

with probability at least 0.9.

## 5.6 Proof of Theorem 4

For  $S = S_{Had}$  we use existing results in Drineas et al. [8] to lower bound  $\alpha^2(S_{Had})$  and  $\beta(S_{Had})$  and then upper bound  $\gamma(S_{Had})$ . Using Lemma 4 in Drineas et al. [8] provides the desired lower bound on  $\alpha^2(S_{Had})$ .

To upper bound  $\beta(S_{Had})$ , we use Lemma 5 in Drineas et al. [8] which states that:

$$\|U^T S_{Had}^T S_{Had} \epsilon\|_2^2 \leq \frac{20d \log(40nd) \|\epsilon\|_2^2}{r},$$

with probability at least 0.9.

Finally to bound  $\gamma(S_{Had})$  recall that  $S_{Had} = S_{Unif} H D$ , where  $S_{Unif}$  is the uniform sampling matrix,

$D$  is a diagonal matrix with  $\pm 1$  entries and  $H$  is the Hadamard matrix:

$$\begin{aligned}
\frac{1}{p}[\|U^T S_{Had}^T S_{Had}\|_F^2] &= \frac{1}{p}[\text{trace}(U^T (S_{Had}^T S_{Had})^2 U)] \\
&= \frac{1}{p}[\text{trace}(U^T (D^T H^T S_{Unif}^T S_{Unif} H D)^2 U)] \\
&= \frac{1}{p}[\text{trace}(U^T D^T H^T S_{Unif}^T S_{Unif} H D D^T H^T S_{Unif}^T S_{Unif} H D U)] \\
&= \frac{1}{p}[\text{trace}(U^T D^T H^T (S_{Unif}^T S_{Unif})^2 H D U)] \\
&= \frac{1}{p} \sum_{j=1}^p \sum_{i=1}^n \sum_{k=1}^n [H D U]_{ij} [H D U]_{kj} [(S_{Unif}^T S_{Unif})^2]_{ki} \\
&= \frac{1}{p} \sum_{j=1}^p \sum_{i=1}^n [H D U]_{ij}^2 [S_{Unif}^T S_{Unif}]_{ii}^2.
\end{aligned}$$

Using Lemma 3 in Drineas et al. [8], with probability greater than 0.95,

$$\frac{1}{p} \sum_{j=1}^p [H D U]_{ij}^2 \leq \frac{2 \log(40np)}{n}.$$

In addition, we have that

$$\begin{aligned}
\frac{1}{p}[\|U^T S_{Had}^T S_{Had}\|_F^2] &= \frac{1}{p} \sum_{j=1}^p \sum_{i=1}^n [H D U]_{ij}^2 [S_{Unif}^T S_{Unif}]_{ii}^2 \\
&\leq \frac{2 \log(40np)}{n} \sum_{i=1}^n [S_{Unif}^T S_{Unif}]_{ii}^2.
\end{aligned}$$

Now we bound  $\mathbb{E}[\sum_{i=1}^n [S_{Unif}^T S_{Unif}]_{ii}^2]$ .

$$\begin{aligned}
\sum_{i=1}^n \mathbb{E}([S_{Unif}^T S_{Unif}]_{ii}^2) &= \frac{n^2}{r^2} \sum_{i=1}^n \sum_{m=1}^r \sum_{\ell=1}^r \mathbb{E}[\sigma_{mi}^2 \sigma_{\ell i}^2] \\
&= \frac{n^2}{r^2} \sum_{i=1}^n \sum_{m=1}^r \sum_{\ell=1}^r \mathbb{E}[\sigma_{mi} \sigma_{\ell i}] \\
&= \frac{n^2}{r^2} \sum_{i=1}^n \left[ \frac{r^2 - r}{n^2} + \frac{r}{n} \right] \\
&= \frac{n^2}{r^2} \left[ \frac{r^2 - r}{n} + r \right] \\
&= n - \frac{n}{r} + \frac{n^2}{r} \\
&\leq n + \frac{n^2}{r}.
\end{aligned}$$



Using Markov’s inequality, with probability greater than 0.9,

$$\sum_{i=1}^n [S_{Unif}^T S_{Unif}]_{ii}^2 \leq 10(n + \frac{n^2}{r}),$$

which completes the proof.

## 6 Discussion and conclusion

In this paper, we developed a framework for analyzing algorithmic and statistical criteria for general sketching matrices  $S \in \mathbb{R}^{r \times n}$  applied to the least-squares objective. Our framework reveals that the algorithmic and statistical criteria depend on different properties of the oblique projection matrix  $\Pi_S^U = U(SU)^\dagger U$ , where  $U$  is the left singular matrix for  $X$ . In particular, the algorithmic criteria depends on the quantity  $\sup_{U^T \epsilon=0} \frac{\|\Pi_S^U \epsilon\|_2}{\|\epsilon\|_2}$ , since in that case the data may be arbitrary and worst-case, whereas the statistical criteria depends on  $\|\Pi_S^U\|_F$ , since in that case the data follow a linear model with homogenous noise variance.

Using our developed framework, we analyzed the statistical and algorithmic performance for 4 sketching schemes, two leverage-score sampling based approaches and two projection-based approaches. In addition to highlighting similarities between sampling and projection algorithms, our theoretical results suggest that leverage-score sampling based approaches perform as well as and sometimes better than projection-based approaches in the regime  $p \log p < r < n$ . This further supports the ideas developed in [21] that using leverage-score sampling based approaches has potential statistical and algorithmic benefits. Our simulation results reveal that for when  $r$  is very close to  $p$ , projection-based approaches tend to out-perform sampling-based approaches since projection-based approaches tend to be more stable in that regime. However when  $r$  is significantly larger than  $p$  but still significantly smaller than  $n$ , sampling-based approaches tend to out-perform projection-based approaches, which is consistent with our theory.

There are numerous ways in which the framework and results from this paper can be extended. Firstly, there is a large literature that presents a number of different approaches to sketching. Since our framework provides general conditions to assess the statistical and algorithmic performance for sketching matrices, a natural and straightforward extension would be to use our framework to compare other sketching matrices. Another natural extension is to determine whether aspects of the framework can be adapted to other statistical models and problems of interest (e.g., generalized linear models, covariance estimation, PCA etc.). Finally, another important direction is to compare the stability and robustness properties of different sketching matrices. Our current analysis assumes a known linear model and it is unclear how the sketching matrices behave under model mis-specification.

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